AN INVESTIGATION OF TWO SEMI-LINEAR PROBLEMS OF OPTIMUM CONTROL

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ABSTRACT

Two problems are presented which are linear on two adjacent intervals but not on their union. These problems are associated with the differential equation $\dot{X} = \begin{cases} AX + BF, & 0 < t < t_1 \\ CX + DF, & t_1 < t < T \end{cases}$, where X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, where F is a 2 x 1 matrix, and where A,B,C, and D are 2 x 2 matrices of functions of t. t_1 is a variable, hence the differential equation is non-linear. Problems associated with this differential equation are called <u>semi</u>-<u>linear</u>.

In the first problem, a condition is found on t_1 and F which must be satisfied whenever x(T) is to be a maximum with y(T) fixed. In the second problem, conditions on F and t_1 are found which must be satisfied for x(T) to be a maximum for a fixed y(T) and for a fixed $x(t_1)$. A numerical routine is developed which yields successive approximations to the maximum value of x(T).

The basic theory of the methods is presented, and the problems are developed in the context of optimum control.

TABLE OF CONTENTS

I. Introduction.

Nonlinearities make the solution of optimum control problems more difficult. In general, different techniques must be developed for each new nonlinear problem.

The particular problems to be discussed are those in which a set of functions x_i of a variable t is related to a second set of functions f_i of t through a differential equation which is linear on each of the intervals $(0,t_1)$ and (t_1,T). If t_1 is a variable, the differential equation, though linear on each of the intervals $(0,t_1)$ and (t_1,T), is non-linear on the interval (0,T). Such a differential equation will be called a <u>semi-linear</u> equation; problems in which a semi-linear differential equation occurs will be called semi-linear problems. Such problems arise in the theory of optimum control; the problems to be discussed will be in the terminology of optimum control theory.

The specific semi-linear problems to be considered arise from the differential equation $\dot{X} = \begin{cases} AX + BF, & 0 < t < t_1 \\ CX + DF, & t_1 < t < T \end{cases}$, where X,A,B,C,D, and F are described as follows: X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ whose elements are continuous functions of t; $\dot{X}$ is the matrix $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$ of derivatives of x and y with respect to t. A,B,C, and D are 2 x 2 matrices whose elements are functions of t which are piecewise continuous and bounded on the interval (0,T). In addition it is necessary that B and D be nonsingular. F is the matrix $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ of functions of t; associated with F there is some constraint, specified in each problem. F is called the matrix of <u>control variables</u>, and those matrices F meeting the specified constraints are called <u>allowable</u>.

Associated with the above differential equation are certain boundary conditions which X(t) is to satisfy. Generally the initial point is

given. In addition, one or more components of X may be specified at some value of t on the interval (0,T). Curves satisfying the given boundary conditions and for which F is allowable, are called <u>admissible</u>.

In the terms that have been defined above, and for the above differential equation, the semi-linear problems to be solved are the following: (1) Find conditions on F and on t_1 which must be satisfied if x(T) is to be a maximum for a given y(T), where t_1 must be determined and where $x(t_1)$ and $y(t_1)$ are unspecified. (2) Add the constraint that $x(t_1)$ is to be specified, for a value of t_1 to be determined. Then (a) find conditions on F and t_1 which must be satisfied for a curve to be admissible; (b) find additional conditions on F and on t_1 such that an admissible curve will yield the maximum x(T); and, (c) develop a method of successive approximations which will give values of x(T) converging to this maximum.

The following results are obtained: In the first problem, a necessary condition at t_1 is derived. In the second problem, conditions for a maximum are derived, and a numerical method is given for obtaining this maximum. These results are another step in the solution of nonlinear problems of optimum control.

I wish to thank Professor Faulkner for the encouragement and help he has given me and for the permission to use class notes of his course, Differential Equations of Optimum Control.

II. General terminology and some sufficient conditions for the linear problem.

In this section we define the linear problem and explain the terminology to be used. We then derive some sufficient conditions for the linear problem.

Let us consider the differential equation

$$(1) \qquad \dot{X} = AX + BF$$

where X is the n x 1 matrix (x_i), where A and B are the n x n matrices (a_{ij}) and (b_{ij}), respectively, and where F is the n x 1 matrix (f_i). We will always assume that B is nonsingular, although the results apply to many problems where B is singular. The a_{ij}, the b_{ij}, and the f_i are functions of t that are bounded and piecewise continuous on the interval (0, T). The a_{ij} and the b_{ij} are given functions; the f_i are functions satisfying a given constraint but otherwise unspecified. The f_i are often called _control variables_ and the x_i _state_, or _dependent variables_.

Associated with the differential equation (1) are certain boundary conditions which we want X(t) to satisfy. Generally the initial point is given. In addition we may specify that certain elements of X take on stated values at various fixed points of the interval (0, T). Note that the differential equation is _linear_, since A and B are functions of t only.

If we multiply both sides of equation (1) on the left by the 1 x n matrix K^T (the transpose of the matrix K), whose elements k^i are functions which have not yet been specified, and integrate from t=0 to t=T, we get

$$(2) \qquad \int_0^T K^T\dot{X}\,dt = \int_0^T K^T AX\,dt + \int_0^T K^T BF\,dt.$$

Integrating the left side by parts and rearranging terms, we get

$$(3)\qquad K^TX\Big|_0^T = \int_0^T(\dot{K}^T + K^TA)X\,dt + \int_0^T K^TBF\,dt.$$

Now let us choose $\dot{K}^T = -K^TA$ and take the transpose; we get

$$(4)\qquad \dot{K} = -A^TK.$$

This is the <u>adjoint equation</u> associated with equation (1). Choosing K as a solution to the adjoint equation and substituting into equation (3), we get

$$(5)\qquad K^TX\Big|_0^T = \int_0^T K^TBF\,dt.$$

We can choose a solution K^1 to the adjoint equation so that $K^{1T}(T)$ is the matrix $(1\ \ 0\ \ 0\ \ \ldots\ 0)$. If we substitute this solution into equation (5) and rearrange terms, we get a solution for $x_1(T)$, namely

$$(6)\qquad x_1(T) = K^{1T}(0)X(0) + \int_0^T K^{1T}(t)BF\,dt.$$

In the same way we chose K^1, we can choose K^2, K^3, ... , K^n such that $K^{iT}(T) = (\delta_{1i}\,\delta_{2i}\,\ldots\,\delta_{ni})$, where δ_{ij} equals zero, if $i\neq j$, or one, if $i=j$. Suppose now that we have the n linearly independent column matrices K^i chosen above. Such a set of n linearly independent matrices is called a <u>fundamental set of solutions</u> for equation (4). If we form the n x n matrix $\mathcal{K}$ whose i'th column is the matrix K^i, then we may write every solution to equation (4) in the form $\mathcal{K}C$, where C is an n x 1 matrix of constants. Conversely, every product of the form $\mathcal{K}C$, being a linear combination of solutions to equation (4), is also a solution. Furthermore $\mathcal{K}$ is itself a solution. This is shown as follows: Take any product $\mathcal{K}C$, where C is an n x 1 matrix of constants and where $\mathcal{K}$ is the matrix defined above. This product is a solution

to equation (4). Hence $\dot{\mathcal{K}}C = -A^T\mathcal{K}C$. But this equation is valid for every C and hence for the C whose transpose is (1 0 0 ...0). Substituting this C into the above equation, we see that the first column of $\dot{\mathcal{K}}$ is the same as the first column of $-A^T\mathcal{K}$. But we could have equally well chosen the i'th element of C as one with the others zero; we hence could have found that the i'th columns of both sides were equal, for $i=1, 2, \ldots, n$. Hence $\dot{\mathcal{K}} = -A^T\mathcal{K}$, i.e. $\mathcal{K}$ is a solution to the adjoint equation, which is what we wanted to show. Furthermore, if $\delta\mathcal{K}$ is a small variation in $\mathcal{K}$, and if $\delta\dot{\mathcal{K}}$ is the corresponding variation in $\dot{\mathcal{K}}$, then $\dot{\mathcal{K}} + \delta\dot{\mathcal{K}} = -A^T(\mathcal{K} + \delta\mathcal{K})$, i.e. $\delta\dot{\mathcal{K}} = -A^T\delta\mathcal{K}$, since $\dot{\mathcal{K}} = -A^T\mathcal{K}$. Hence the variations in $\mathcal{K}$ are related to the variations in $\dot{\mathcal{K}}$ by the adjoint equation.

We have shown that solutions to the adjoint equation can be used to find solutions to the equation $\dot{X} = AX + BF$ when F is known. The next problem we are concerned with is that of finding F so that for a given X(0) and a given T, a specified component of X(T) is a maximum. F can be regarded as a column matrix of forcing functions; these forcing functions are the components of the <u>forcing function vector</u>. There may be one of several types of constraints on F. In rocket thrust scheduling, for example, the acceleration possible at any one time is limited by a function of the mass of fuel on board at that time. If we call this function $\varphi(t)$, the corresponding constraint on F is that $|F| \leq \varphi(t)$, where $|F|$ is the square root of the sum of the squares of the elements of F. Another type of constraint on F is $|f_1| \leq |a_i|$, where a_i is a given function of t. Problems of this second kind are called <u>bang-bang</u> control problems. [1] In every problem it is necessary to state the constraint on F; forcing functions which satisfy the stated constraints will be called allowable. Solution curves to equation (1) for which F is allowable will be called <u>allowable curves</u>; allowable

curves satisfying the differential equation and satisfying the given boundary conditions will be called admissible.

We want a method for choosing an allowable F such that X(0) is a given point and such that a specified element of X(T) is a maximum. A most important principle enables us to state conditions which F must satisfy whenever the desired maximization takes place, namely Pontryagin's Maximum Principle: The control variables must be chosen from the set of allowable controls so as to maximize a scalar product of some solution to the adjoint equation and the forcing function vector at every time t. [2] We will use this principle extensively in the following pages.

Consider the following problem and see how the maximum principle may apply to it. Suppose that we want a curve beginning at some fixed point at time t = 0 on which some element of X at t=T, say $x_1(T)$, is a maximum and such that on the curve these three conditions hold: First, F is allowable. Second, the curve satisfies the differential equation $\dot{X} = AX + BF$ for all values of t between t=0 and t=T where $T > 0$ is given. Third, $x_i(T) = x_{iT}$, for i=2, ... , N, where x_{iT} are given, and where $1 \leq N \leq n$, and where $x_{N+1}(T), \ldots , x_n(T)$ are unspecified. N=1 means that no values of the x_i are given at t=T; N=n means that all values except x_1 are given at t=T. A proof given by Faulkner [3] proves that the following hypotheses are sufficient for a curve C* to yield a maximum $x_1(T)$. Suppose that we have found a curve C* with forcing function F^* and have found at the same time a solution K^* to the adjoint equation which together satisfy the following hypotheses:

H1. C^* is admissible: it begins at X_o and ends on the manifold defined by $x_2(T) = x_{2T}, \ldots , x_N(T) = x_{NT}$, and F^* is allowable.

H2. F^* maximizes $K^{*T}BF$ for all allowable F, i.e. $K^{*T}BF^* \geqslant K^{*T}BF$ for all allowable F.

H3. $k_1^*(T) > 0$, and $k_i^*(T) = 0$ for $i > N$. No restriction is put on $k_i^*(T)$ for $i = 2, \dots, N$.

<u>THEOREM</u>: C^* furnishes the desired maximum of $x_1(T)$.

<u>Proof</u>: We have shown that $K^TX\Big|_0^T = \int_0^T K^TBF\,dt$ for every solution K to the adjoint equation. Hence for the particular solution K^* and the matrix X^* we have, on the path C^*,

$$(7)\qquad k_1^*(T)x_1^*(T) + k_2^*(T)x_{2T} + \dots + k_N^*(T)x_{NT} = (K^{*T}X^*)_T$$

$$= (K^{*T}X^*)_0 + \int_0^T K^{*T}BF\,dt.$$

Consider any other admissible path C' with $F = F'$. For this path and for $K = K^*$ we have

$$(8)\qquad k_1^*(T)x_1'(T) + \dots + k_N^*(T)x_{NT} = (K^{*T}X^*)_0 + \int_0^T K^{*T}BF'\,dt.$$

Subtracting equation (8) from equation (7), we get

$$(9)\qquad k_1^*(x_1^*-x_1')_T = \int_0^T (K^{*T}BF^*-K^{*T}BF')\,dt \geqslant 0.$$

Hence, since $k_1^*(T) > 0$, $x_1^*(T) \geqslant x_1'$ at $t = T$. Hence the given hypotheses are indeed sufficient to give a maximum $x_1(T)$. Note that since $k_j^*(T) = 0$, for $j=N+1, \dots, n$, the unspecified elements of $X^*(T)$ play no part in the solution.

Having developed a useful sufficiency condition, let us now consider the case where X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, where A and B are 2 x 2 matrices, where F is the matrix $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, and where we want to maximize x(T) subject to the following constraints: First, $(f_1)^2 + (f_2)^2 = 1$. Second, $T > 0$ is

fixed. Third, solution curves must start at X_o and end on the line $y(T) = y_f$, where y_f is given. For this problem the admissible curves are allowable curves satisfying the third constraint.

Let us choose the solutions K^1 and K^2 to the adjoint equation having $K^1(T) = \begin{pmatrix}1\\0\end{pmatrix}$ and $K^2(T) = \begin{pmatrix}0\\1\end{pmatrix}$. Using these solutions in equation (6), we get

$$x(T) = (K^{1T}X)_0 + \int_0^T K^{1T}BF\,dt \tag{10}$$

and

$$y(T) = (K^{2T}X)_0 + \int_0^T K^{2T}BF\,dt. \tag{11}$$

Since the given constraint on F is that $(f_1)^2+(f_2)^2 = 1$, we may choose $f_1 = \cos\theta$, $f_2 = \sin\theta$, where θ is a function of t. With this substitution, a variation in θ will cause a variation in F which will in turn cause a variation in x and in y. Let us call the variation in θ, $\delta\theta$ and the variations in x and in y at t=T, δx and δy, respectively. We then get, from equations (10) and (11),

$$\delta x = \int_0^T K^{1T}B\frac{\partial F}{\partial\theta}\,\delta\theta\,dt \tag{12}$$

and

$$\delta y = \int_0^T K^{2T}B\frac{\partial F}{\partial\theta}\,\delta\theta\,dt, \tag{13}$$

for sufficiently small values of $\delta\theta$. Let us now choose a special variation of the form $\theta = e_1K^{1T}B\frac{\partial F}{\partial\theta} + e_2K^{2T}B\frac{\partial F}{\partial\theta}$ where e_1 and e_2 are constants yet to be chosen. Substituting this variation in equation (12), we get

$$\delta x = \int_0^T (K^{1T}B\frac{\partial F}{\partial\theta})(e_1K^{1T}B\frac{\partial F}{\partial\theta} + e_2K^{2T}B\frac{\partial F}{\partial\theta})dt \tag{14}$$

and a similar expression for δy. These can be simplified if we let

$$I^{ij} = \int_0^T (K^{iT}B\frac{\partial F}{\partial \theta})(K^{jT}B\frac{\partial F}{\partial \theta})dt.$$

Using this substitution we get

(15) $$\delta x = e_1 I^{11} + e_2 I^{12}$$

and

(16) $$\delta y = e_1 I^{21} + e_2 I^{22}.$$

Note first that $I^{12} = I^{21}$. Note next the consequences if $I^{22} = 0$. If $I^{22} = 0$, then $K^{2T}B\frac{\partial F}{\partial \theta}$ must be identically zero; hence I^{21} and δy are both identically zero; hence $y(T)$ is not affected by a change in the control variable θ. We describe such a situation by saying that the curve furnishes a stationary value for $y(T)$. We will not want y to have a stationary value, and we shall henceforth assume for our curve that $I^{22} \neq 0$, or, equivalently, that $K^{2T}B\frac{\partial F}{\partial \theta}$ is not identically zero on the interval $(0,T)$.

Since T is fixed, equations (15) and (16) give the total differentials of x and of y. If we replace δy by $y_f - y(T)$, we can generally find values for e_1 and e_2 which give a variation which will make the resulting curve admissible.

Suppose now that we have an admissible curve and that we want to find conditions on the I^{ij} such that $x(T)$ is a maximum. For an admissible curve we have

(17) $$\delta x = e_1 I^{11} + e_2 I^{12}$$

and

(18) $$0 = e_1 I^{21} + e_2 I^{22}.$$

Equations (17) and (18) can be solved for $\delta x > 0$ only if the rank of

$\begin{pmatrix} \delta x & I^{11} & I^{12} \\ 0 & I^{21} & I^{22} \end{pmatrix}$ is equal to the rank of $\begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix}$. But the rank of the first matrix is the same as the rank of $\begin{pmatrix} \delta x & 0 & 0 \\ 0 & I^{21} & I^{22} \end{pmatrix}$. The rank of this last matrix is one greater than the rank of the matrix $(I^{21} \; I^{22})$; hence equations (17) and (18) cannot be solved for δx whenever the rank of $\begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix}$ is equal to the rank of $(I^{21} \; I^{22})$. But this will be true only when the determinant $\left| I^{ij} \right|$ is zero. Hence if the admissible curve yields a maximum for x(T), then $\left| I^{ij} \right| = 0$.

Let us consider some of the consequences of this condition. If the determinant $\left| I^{ij} \right| = 0$, then the matrix (I^{ij}) has a zero eigenvalue. Let us choose constants c_1 and c_2 such that $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an eigenvector of (I^{ij}) corresponding to this zero eigenvalue. Then let us consider

$$(19) \qquad J = \int_0^T (c_1 K^{1T} B \frac{\partial F}{\partial \theta} + c_2 K^{2T} B \frac{\partial F}{\partial \theta})^2 \, dt.$$

Clearly $J \geq 0$. But $J = (c_1)^2 I^{11} + 2c_1 c_2 I^{12} + (c_2)^2 I^{22}$

$$= (c_1 c_2) \begin{pmatrix} I^{11} & I^{12} \\ I^{21} & I^{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

But $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an eigenvector corresponding to the zero eigenvalue of (I^{ij}), hence $J = 0$. Therefore $c_1 K^{1T} B \frac{\partial F}{\partial \theta} + c_2 K^{2T} B \frac{\partial F}{\partial \theta} = 0$. This equation appears frequently in the literature on calculus of variations - it is known as the Euler equation. Curves whereon the Euler equation is satisfied are frequently called extremals.[1] Hence $K^* = c_1 K^1 + c_2 K^2$ is a

[1] This is the definition given by Bolza [4]. Another definition is given by Bliss [5].

solution to the adjoint equation which gives an extremal.

We saw that if $c_1 > 0$ and if $\theta = \theta^*$ maximizes $K^{*T}BF$ then $x(T)$ is a maximum. The condition $K^{*T}B\frac{\partial F}{\partial \theta} = 0$ is a necessary condition for $x(T)$ to be a maximum. We can and will choose $c_1 > 0$. We will also see that it is necessary that θ maximize $K^{*T}BF$; this is the Weierstrass condition. In this problem the Weierstrass condition is necessary and sufficient for an admissible curve to furnish the desired maximum.

Let us assume that the rank of (I^{ij}) is at least one for all curves which arise: this is called <u>normality</u> for the problem. Suppose now that we have an admissible, normal curve which furnishes the desired maximum. On it the Euler equation must be satisfied, and the matrix (I^{ij}) has a zero eigenvalue with a corresponding eigenvector $\binom{c_1}{c_2}$; we will choose $c_1 > 0$. Then the F determined by the Euler equation maximizes the product $K^{*T}BF$, as a function of θ, over the entire interval from $t = 0$ to $t=T$. The proof is as follows:

Suppose that $\theta_1(t)$ is the argument of F for some admissible curve satisfying the Euler equation but that $\theta_1(t)$ does not maximize the product $K^{*T}BF(\theta)$ on some subinterval (t_1,t_2) of $(0,T)$. Then there is some other θ, say $\theta_2(t)$, such that $K^{*T}BF(\theta_2) > K^{*T}BF(\theta_1)$ on the interval (t_1, t_2). Let $\theta = \theta_2$ on the interval $(t_1,t_1 + dt_1)$, for $t_1 + dt_1 < t_2$, and let $\delta\theta = e_1K^{1T}B\frac{\partial F}{\partial \theta} + e_2K^{2T}B\frac{\partial F}{\partial \theta}$ elsewhere on the interval $(0,T)$. Then, since both paths are admissible,

$$(20) \qquad dx = K^{1T}B\left[F(\theta_2) - F(\theta_1)\right] dt_1 + I^{11}e_1 + I^{12}e_2$$

and

$$(21) \qquad 0 = K^{2T}B\left[F(\theta_2) - F(\theta_1)\right] dt_1 + I^{21}e_1 + I^{22}e_2.$$

If we now multiply equation (20) by c_1 and (21) by c_2 and add, remembering that $K^{*T} = c_1K^{1T} + c_2K^{2T}$, we get

$$(22) \qquad c_1 dx = K^{*T}B\left[F(\theta_2) - F(\theta_1)\right] dt_1$$

since $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ is an eigenvector of (I^{ij}) corresponding to its zero eigenvalue. Since the right-hand side of equation (22) and the constant c_1 are both positive, the dx of equation (22) is positive. Furthermore it is possible to satisfy equations (20) and (21) by a set of e_i which will give a positive dx and at the same time keep dy = 0. To see this, multiply equation (20) by c_1 and call the new equation (20'); multiply equation (21) by c_2 and call the new equation (21'). Add equation (20') to equation (21') and call the resulting equation (23). Looking now at equations (23) and (20), remembering that $K^{*T} = c_1K^{1T} + c_2K^{2T}$, we have

$$(23) \qquad c_1 dx = K^{*T}B\left[F(\theta_2) - F(\theta_1)\right] dt_1$$

and

$$(20) \qquad 0 = K^{2T}B\left[F(\theta_2) - F(\theta_1)\right] dt_1 + I^{21}e_1 + I^{22}e_2.$$

But, by assumption, $K^{*T}B\left[F(\theta_2) - F(\theta_1)\right] > 0$, and $I^{22} \neq 0$, hence we can solve these two equations for e_1 and e_2. Hence it is possible to satisfy equations (20) and (21) by a set of e_i which will give a variation which will in turn yield a larger value of x(T). Hence if $\theta(t)$ is such that the product $K^{*T}BF(\theta)$ is not a maximum on every subinterval of $(0,T)$, then x(T) will not be a maximum. It is worthy of note that the form of the conditions for an extremum is independent of the particular problem.

The Weierstrass-Erdmann corner condition is an immediate consequence of the above condition. For suppose that we have a curve whereon the hypotheses of the above theorem are satisfied. Suppose further that t_1 is a point at which the control variable $\theta(t)$ is discontinuous. Then $(c_1K^{1T} + c_2K^{2T})BF(\theta) \Big|_{t_{1-}}^{t_{1+}} = 0$. Proof: This must be so; otherwise the condition just found would not be satisfied in some neighborhood of t_1,

for either $t > t_1$ or $t < t_1$. The condition that $K^{*T}BF(\theta)\big|_{t_{1-}} = K^{*T}BF(\theta)\big|_{t_{1+}}$ is commonly known as the Weierstrass-Erdmann Corner Condition.

In this section we have introduced several important concepts as they pertain to a linear problem discussed in the following sections.

In the next section we consider a combination of linear systems, the combination being non-linear.

III. The Weierstrass-Erdmann corner condition for a more general problem.

In this section we consider a problem having a corner at a variable time. We develop the Weierstrass-Erdmann corner condition for this particular problem.

Let us consider the differential equation

$$(24) \qquad \dot{X} = A'X + B'F$$

where A' is A, for $0 < t < t_1$, and C, for $t_1 < t < T$, where B' is B, for $0 < t < t_1$, and D, for $t_1 < t < T$ X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$; F is the matrix $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$; A,B,C, and D are 2 x 2 matrices of functions of t which are piecewise continuous and bounded on their respective intervals; B and D are non-singular. T is fixed; t_1 is a variable to be determined. The constraint on F is as before, namely $(f_1)^2 + (f_2)^2 = 1$. In addition, x and y must be continuous at t_1. For this problem the admissible curves are the allowable curves starting at X_o and ending on the line $y(T) = y_f$.

Note that this problem is non-linear on the interval (0,T), since A' and B' are functions of t_1 as well as of t. The problem is, however, linear on each of the two sub-intervals $(0,t_1)$ and (t_1,T). Hence we call this a <u>semi-linear</u> problem.

We can rewrite equation (24) as

$$(25) \qquad \dot{X} = \begin{cases} AX + BF, & 0 < t < t_1 \\ CX + DF, & t_1 < t < T. \end{cases}$$

The adjoint equation is

$$(26) \qquad \dot{K} = \begin{cases} -A^T K, & 0 < t < t_1 \\ -C^T K, & t_1 < t < T. \end{cases}$$

We choose, as particular solutions to $\dot{K} = -C^T K$, the solutions $K^1(T) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $K^2(T) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. For $t_1 < t < T$, then, each K^i is a function of T and of t. Since T is to remain fixed, however, we shall suppress it wherever it

occurs in the K^i and shall consider the K^i as functions of t only, on the interval (t_1,T). Let us define a set of particular solutions to the equation $\dot{K} = A^TK$ by $K^i(t_{1-}) = K^i(t_{1+})$, for i = 1,2. For t on the interval $(0,t_1)$, then, the K^i are functions of the variables t_1 and t where, as above, we suppress T.

Before continuing further let us make one substitution. When convenient, we shall use E to mean B, in the interval $(0,t_1)$ and D, in the interval (t_1,T). (Note that E and B' are the same matrix). Using this convention, we get

$$(27) \qquad x(T) = K^{1T}(t_1,0)X(0) + \int_0^T K^{1T}EF\, dt$$

and

$$(28) \qquad y(T) = K^{2T}(t_1,0)X(0) + \int_0^T K^{2T}EF\, dt.$$

Since the constraint on F is that $(f_1)^2 + (f_2)^2 = 1$, we can replace f_1 by $\cos\theta$ and f_2 by $\sin\theta$, where θ is a function of t. Since we are considering both θ and t_1 as variable, we get, from equations (27) and (28),

$$(29) \qquad dx(T) = \frac{\partial x(T)}{\partial t_1} dt_1 + \delta x(T)$$

and

$$(30) \qquad dy(T) = \frac{\partial y(T)}{\partial t_1} dt_1 + \delta y(T)$$

where $\delta x(T)$ and $\delta y(T)$ are the variations in x(T) and in y(T) due to variations in the control variable θ.

$\delta x(T)$ and $\delta y(T)$ are given by the following equations for sufficiently small values of θ:

$$(31) \qquad \delta x(T) = \int_0^T K^{1T}E\frac{\partial F}{\partial \theta}\,\delta\theta\, dt$$

and

(32) $$\delta y(T) = \int_0^T K^{2T} E \frac{\partial F}{\partial \theta} \delta\theta \, dt$$

Let us choose the special variation $\delta\theta = e_1 K^{1T} E \frac{\partial F}{\partial \theta} + e_2 K^{2T} E \frac{\partial F}{\partial \theta}$ where e_1 and e_2 are constants which remain to be specified. Then equation (31) becomes

(33) $$\delta x(T) = \int_0^T \left(K^{1T} E \frac{\partial F}{\partial \theta}\right)\left(e_1 K^{1T} E \frac{\partial F}{\partial \theta} + e_2 K^{2T} E \frac{\partial F}{\partial \theta}\right) dt$$

with a similar expression for $\delta y(T)$. Adopting the substitution

(34) $$I^{ij} = \int_0^T \left(K^{iT} E \frac{\partial F}{\partial \theta}\right)\left(K^{jT} E \frac{\partial F}{\partial \theta}\right) dt,$$

we get

(35) $$\delta x(T) = e_1 I^{11} + e_2 I^{12}$$

and

(36) $$\delta y(T) = e_1 I^{21} + e_2 I^{22}.$$

We will use equations (35) and (36) later.

The variations in $x(T)$ and in $y(T)$ due to a variation in t_1 are more complicated. Rewriting equations (27) and (28) to show the way in which t_1 enters, we get

(37) $$x(T) = K^{1T}(t_1,0)X(0) + \int_0^{t_1} K^{1T}(t_1,t)BF dt + \int_{t_1}^T K^{1T}(t)DF \, dt$$

and

(38) $$y(T) = K^{2T}(t_1,0)X(0) + \int_0^{t_1} K^{2T}(t_1,t)BF dt + \int_{t_1}^T K^{2T}(t)DF \, dt.$$

Hence

(39) $$\frac{\partial x(T)}{\partial t_1} dt_1 = \frac{\partial K^{1T}}{\partial t_1}(t_1,0)X(0)dt_1 + \int_0^{t_1} \frac{\partial K^{1T}}{\partial t_1}(t_1,t)BF dt_1 dt$$

$$+ \left[(K^{1T}BF)_{t_{1-}} - (K^{1T}DF)_{t_{1+}} \right] dt_1$$

and a similar expression for $\frac{\partial y(T)}{\partial t_1}$, the only difference being that

each K^{1T} is replaced by K^{2T}. To avoid problems in notation let us denote by δK^{1T} the variation in K^{1T} caused by varying t_1 by a small amount dt_1, i.e. to a first-order approximation $\delta K^{1T} = \frac{\partial K^{1T}}{\partial t_1} dt_1$. With this notation, equation (39) becomes

$$(40) \qquad \frac{\partial x(T)}{\partial t_1} dt_1 = \delta K^{1T}(t_1,0)X(0) + \int_0^{t_1} \delta K^{1T}(t_1,t)BF\, dt$$

$$+ \left[(K^{1T}BF)_{t_{1-}} - (K^{1T}DF)_{t_{1+}}\right] dt_1 .$$

Since K^1 is not a function of t_1 on the interval (t_1,T), $\delta K^1 = 0$ on that interval. For t on the interval $(0,t_1)$, δK^1 satisfies the adjoint equation, and hence $\delta \dot{K}^1 = -A^T\delta K^1$ on $(0,t_1)$. Keeping this in mind, let us consider the integral in equation (40). From equation (25), $BF = \dot{X} - AX$. With this substitution, the integral in equation (40) becomes

$$(41) \qquad \int_0^{t_1} \delta K^{1T}(t_1,t)\dot{X}\, dt - \int_0^{t_1} \delta K^{1T}(t_1,t)AX\, dt.$$

Integrating the first integral by parts, we get

$$(42) \qquad \delta K^{1T}(t_1,t)X(t) \Big|_0^{t_1} - \int_0^{t_1} (\delta \dot{K}^{1T} + \delta K^{1T}A)X\, dt.$$

But δK^1 satisfies the adjoint equation, hence the integral in (42) is zero. Hence (42) reduces to $\delta K^{1T}(t_1,t_1)X(t_1) - \delta K^{1T}(t_1,0)X(0)$. But the variation in K^{1T} due to a variation in t_1, evaluated at t_1, is equal to $\left[K^{1T}(t_{1-})A - K^{1T}(t_{1+})C\right] dt_1$, by equation (25). Furthermore, X(t) is continuous, so that $X(t_{1-}) = X(t_{1+})$. Hence (42) is equal to

$\left[K^{1T}(t_{1-})A - K^{1T}(t_{1+})C\right] X(t_1)\, dt_1 - \delta K^{1T}(t_1,0)X(0)$, and equation (40) reduces to

$$(43) \qquad \frac{\partial x(T)}{\partial t_1} dt_1 = K^{1T}(t_{1-}) \left[AX(t_1) + BF(t_{1-})\right] dt_1$$

$$- K^{1T}(t_{1+})\left[CX(t_1) + DF(t_{1+})\right] dt_1 .$$

But, by equation (25), this says that

$$(44) \qquad \frac{\partial x(T)}{\partial t_1} dt_1 = \left[K^{1T}(t_{1-})\dot{X}(t_{1-}) - K^{1T}(t_{1+})\dot{X}(t_{1+})\right] dt_1 .$$

Similarly,

$$(45) \qquad \frac{\partial y(T)}{\partial t_1} dt_1 = \left[K^{2T}(t_{1-})\dot{X}(t_{1-}) - K^{2T}(t_{1+})\dot{X}(t_{1+})\right] dt_1 .$$

Hence equations (29) and (30) give, as the total variations in x(T) and y(T),

$$(46) \qquad dx(T) = \int_0^T K^{1T}E\frac{\partial F}{\partial \theta}\,\delta\theta\, dt - \left(K^{1T}\dot{X}\right)\Big|_{t_{1-}}^{t_{1+}} dt_1$$

and

$$(47) \qquad dy(T) = \int_0^T K^{2T}E\frac{\partial F}{\partial \theta}\,\delta\theta\, dt - \left(K^{2T}\dot{X}\right)\Big|_{t_{1-}}^{t_{1+}} dt_1$$

Choose the special variation $\delta\theta = (e_1K^{1T} + e_2{}^{2T})E\frac{\partial F}{\partial \theta}$. Then, using equations (35) and (36), we get

$$(48) \qquad dx(T) = I^{11}e_1 + I^{12}e_2 - (K^{1T}\dot{X})\Big|_{t_{1-}}^{t_{1+}} dt_1$$

$$(49) \qquad dy(T) = I^{21}e_1 + I^{22}e_2 - (K^{2T}\dot{X})\Big|_{t_{1-}}^{t_{1+}} dt_1$$

Observe that if t_1 is fixed we have the problem discussed earlier.

Let us assume, then, that admissible curves exist and that our problem has a solution. If admissible curves exist, it is possible to find e_1, e_2 and dt_1 from equations (48) and (49) and hence to get a variation $\delta\theta$ such that the curve obtained by replacing θ by $\theta + \delta\theta$ is admissible. Suppose that we have performed these calculations and have obtained an admissible curve. For this curve, dy(T) in equation (49) is zero. Before continuing further, let us assume that $I^{22} \neq 0$. This assumption

is enough to insure normality in this problem. Having, then, an admissible normal curve, we can use the results obtained after equation (19).

We showed that on an admissible normal curve for which $x(T)$ is a maximum, the matrix (I^{ij}) has one zero eigenvalue and that the components c_1 and c_2 of the eigenvector can be chosen so that c_1 is positive. We also showed that one solution to the adjoint equation which gives an admissible extremal is $K^* = c_1K^1 + c_2K^2$. Hence let us choose c_1 and c_2 as components of an eigenvector of the matrix (I^{ij}) corresponding to the zero eigenvalue, with $c_1 > 0$. Multiplying equations (48) and (49) by these c_i and adding, remembering that the K^i are continuous at t_1, gives

$$(50) \qquad c_1 dx = K^{*T}(t_1) \left[\dot{X}(t) \Big|_{t_{1-}}^{t_{1+}} dt_1 \right] .$$

This leads to the following theorem:

<u>THEOREM</u>: For $x(T)$ to be a maximum, the quantity $K^{*T}(t_1) \left[\dot{X}(t) \Big|_{t_{1-}}^{t_{1+}} \right]$ must equal zero.

Proof: If the above quantity is positive, any $dt_1 > 0$ will give a positive $dx(T)$ and hence a larger $x(T)$. If the above quantity is negative, any $dt_1 < 0$ will give a positive $dx(T)$. This is the Weierstrass-Erdmann corner condition for this problem.

The next section uses the problem we have been studying for background and considers the essentially different problem obtained by introducing the constraint that $x(t_1)$ has some fixed value. We develop a numerical routine for determining the solution by a method of successive approximations.

IV. A numerical routine for determining the maximum $x(T)$ for a fixed value of $x(t_1)$

In this section we consider the problem states as before but with the different constraint that $x(t_1)$ has some fixed value. We make up a curve consisting of two arcs, each of which is an extremal. We use the method of variation of extremals on these arcs to drive the resulting curve to admissibility and in a gradient technique to determine the curve on which $x(T)$ is a maximum.

Let us again consider equation (24), namely

$$\text{(24)} \qquad \dot{X} = A'X + B'F$$

with the same conditions as in the beginning of Section III but with a different constraint on t_1, namely that $x(t_1) = x_1$, where x_1 is given.

This problem, like the one in the preceding section, is semilinear. It is not linear on the entire interval $(0,T)$, since A' and B' are functions of t_1 as well as of t; it is, however, linear on each of the two sub-intervals $(0,t_1)$ and (t_1,T).

The differential equation we have been working with is

$$\text{(25)} \qquad \dot{X} = \begin{cases} AX + BF, & 0 < t < t_1 \\ CX + DF, & t_1 < t < T. \end{cases}$$

For this problem the admissible curves are allowable curves satisfying the above differential equation and such that $X(0) = X_o$, $x(t_1) = x_1$, and $y(T) = y_f$, where x_1 and y_f are given constants. The adjoint equation for equation (25) is

$$\text{(26)} \qquad \dot{K} = \begin{cases} -A^TK, & 0 < t < t_1 \\ -C^TK, & t_1 < t < T. \end{cases}$$

We choose solutions K^1 and K^2 to the adjoint equation such that $K^1(T) = \begin{pmatrix}1\\0\end{pmatrix}$ and $K^2(T) = \begin{pmatrix}0\\1\end{pmatrix}$ and such that K^i are continuous at $t = t_1$. We further define the 2 x 2 matrix $\mathcal{K}$ as before; its first column is $K^1(t)$ and its

second column is $K^2(t)$. Note that $\mathcal{K}$ is a function of the variable t for $t_1 < t < T$ and of both the variables t_1 and t for $0 < t < t_1$.

Multiplying equation (25) on the left by $\mathcal{K}^T$, integrating from t = 0 to $t = t_1$, and using the fact that $\mathcal{K}$ is a solution to the adjoint equation leads to

$$(51) \qquad \mathcal{K}^T(t_1,t)X(t) \Big|_0^{t_1} = \int_0^{t_1} \mathcal{K}^T(t_1,t)BF\,dt.$$

We have also, from the last section,

$$(37) \qquad x(T) = K^{1T}(t_1,0)X(0) + \int_0^{t_1} K^{1T}(t_1,t)BFdt + \int_{t_1}^{T} K^{1T}(t)DFdt$$

$$(38) \qquad y(T) = K^{2T}(t_1,0)X(0) + \int_0^{t_1} K^{2T}(t_1,t)BFdt + \int_{t_1}^{T} K^{2T}(t)DFdt.$$

The differential equation is linear on each of the intervals.

We choose F by using the method of variation of extremals on $(0,t_1)$ and each of them (t_1,T). On each of the two arcs, we have the following by the theorem of section II. If C^* is an admissible arc, if K^* is the solution to the adjoint equation defined by $K^* = c_1K^1 + c_2K^2$, where K^1 and K^2 are the solutions to the adjoint equation defined above, if $c_1 > 0$, and if, on C^*, F^* maximizes the scalar $K^{*T}EF$, then C^* furnishes a maximum x at the end of the arc, relative to the point at which the arc began. If we, then, choose F^* to maximize the product $K^{*T}BF$, for $0 < t < t_1$, and $K^{*T}DF$, for $t_1 < t < T$, the resulting curve will be made up of an extremal from t = 0 to $t = t_1$ and an extremal from $t = t_1$ to t = T.

Once we have an admissible curve made up of two extremal arcs, we want some way to vary these curves so as to get a maximum x(T). We use a gradient technique, explained later, to choose a set of variations in the curve parameters to get a larger x(T). The new curve may not be ad-

missible because of second-order effects. Hence, we again drive the curve to admissibility, say, by the method of variation of extremals mentioned above.[1] We continue this process until the maximum $x(T)$ is obtained.

The calculations proceed as follows: Since c_1 is positive on each arc, we can choose $c_1 = 1$. Hence $K^* = K^1 + cK^2$, where c is a constant. Unfortunately when a condition on X is given at time t_1, K^* may not be continuous at t_1. Hence let us suppose that c is not the same in both arcs, and let us adopt the following notation: $K^* = K^1 + aK^2$, for t on $(0,t_1)$, $K^* = K^1 + bK^2$, for t on (t_1,T), where a, b, and t_1 are unknowns which must be determined so that the curve made up of the arcs is admissible and yields a maximum $x(T)$. Now consider F^*. F^* maximized $K^{*T}BF$ on the interval $(0,t_1)$; on that interval K^* is a function of t_1, t, and a. On the interval (t_1,T), K^* is a function of b and t only, hence F^* is a function of b and t on the second interval.

With F replaced by F^*, equations (51), (37), and (38) become

$$\mathcal{K}^T(t_1,t)X(T)\Big|_0^{t_1} = \int_0^{t_1} \mathcal{K}^T(t_1,t)BF^*dt \tag{52}$$

$$x(T) = K^{1T}(t_1,0)X(0) + \int_0^{t_1} K^{1T}(t_1,t)BF^*dt + \int_{t_1}^{T} K^{1T}(t)DF^*dt \tag{53}$$

$$y(T) = K^{2T}(t_1,0)X(0) + \int_0^{t_1} K^{2T}(t_1,t)BF^*dt + \int_{t_1}^{T} K^{2T}(t)DF^*dt \tag{54}$$

From these equations we want to devise a routine that will, first, give

[1] It is possible that we might obtain an $x(T)$ from a set of curves that are admissible under one admissibility criterion but not under another. After trying various admissibility criteria, the author decided upon that of calling a curve admissible if $|x(t_1)-x_1| + |y(T)-y_f| \leq 10^{-4}$.

us a set of admissible curves and, second, find an admissible curve whereon $x(T)$ has its maximum value. We want also to find a condition which will indicate that no admissible curves exist, if such is indeed the case. In this problem, a curve can be characterized by a set of values for t_1, a, and b. Hence we want a routine to find values of a, b, and t_1 which will determine an admissible curve whereon $x(T)$ is a maximum.

Let us first get expressions for the total differentials of $x(t_1)$, of $x(T)$, and of $y(T)$. First, of $x(t_1)$. Equation (52) can be written

$$(55) \qquad \mathcal{K}^T(t_1,t_1)X(t_1) = \mathcal{K}^T(t_1,0)X(0) + \int_0^{t_1}\mathcal{K}^T(t_1,t)BF^* dt.$$

Taking differentials of both sides gives

$$(56) \qquad d\,\mathcal{K}^T(t_1,t_1)X(t_1) + \mathcal{K}^T(t_1,t_1)dX(t_1) = d\,\mathcal{K}^T(t_1,0)X(0) + \mathcal{K}^T(t_1,t_1)BF^*(t_{1-})dt_1 + \int_0^{t_1} d\,\mathcal{K}^T(t_1,t)BF^* dt + \int_0^{t_1} \mathcal{K}^T(t_1,t)B\delta F^* \, dt.$$

To get $d\,\mathcal{K}^T(t_1,t_1)$, note that the first t_1 denotes the end of the interval $(0,t_1)$ and that the second is the value that the running variable t assumed at the end of the interval. The differential due to the change in the first t_1 is, since $\mathcal{K}$ satisfies the adjoint equation, $\mathcal{K}^T(t_1,t_1)(A-C)dt_1$; the differential due to the change in the second t_1 is $-\mathcal{K}^T(t_1,t_1)A\,dt_1$. Hence $d\,\mathcal{K}^T(t_1,t_1) = -\mathcal{K}^T(t_1,t_1)Cdt_1$ and $d\,\mathcal{K}^T(t_1,t_1)X(t_1) = -\mathcal{K}^T(t_1,t_1)CX(t_1)dt_1$.

Consider next $\int_0^{t_1} d\,\mathcal{K}^T(t_1,t)BF^* dt$. By the argument after equation (42), this integral is equal to $\mathcal{K}^T(t_1,t_1)(A-C)X(t_1)dt_1 - d\,\mathcal{K}^T(t_1,0)X(0)$. Substituting the above results into equation (56) yields

$$(57)\qquad \mathcal{K}^T(t_1,t_1)dX(t_1) = \mathcal{K}^T(t_1,t_1)AX(t_1)dt_1 + \mathcal{K}^T(t_1,t_1)BF(t_{1-})dt_1 + \int_0^{t_1} \mathcal{K}^T(t_1,t)B\delta F^* dt$$

$$= \mathcal{K}^T(t_1,t_1)\dot{X}(t_{1-})dt_1 + \int_0^{t_1} \mathcal{K}^T(t_1,t)B\delta F^* dt.$$

Calculating the integral in the above equation is somewhat more difficult. F^* is chosen to maximize the produce $(K^{1T} + aK^{2T})BF$. Expanding the product $(K^{1T} + aK^{2T})B$ yields the matrix

$$\left(\left[k^{11}+ak^{12}\right] b_{11} + \left[k^{21}+ak^{22}\right]b_{21} \qquad \left[k^{11}+ak^{12}\right]b_{12}+ \left[k^{21}+ak^{22}\right]b_{22}\right) .$$

Calling the first element h_1 and the second h_2 and remembering both that F^* must maximize the product $(h_1\ h_2)F$ and that F can be written as $\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}$, we see that $\cos\theta = h_1/\left[(h_1)^2 + (h_2)^2\right]^{\frac{1}{2}}$ and that $\sin\theta = h_2/\left[(h_1)^2 + (h_2)^2\right]^{\frac{1}{2}}$. Hence $\delta F^* = \begin{pmatrix}-\sin\theta \\ \cos\theta\end{pmatrix}\delta\theta$, where $\theta = \tan^{-1}(h_2/h_1)$. Substituting in the expressions for $\sin\theta$ and for $\cos\theta$ and finding $\delta\theta$ in terms of δh_1 and δh_2 yields

$$\delta F^* = \begin{pmatrix}-h_2 \\ h_1\end{pmatrix}\frac{h_1\delta h_2 - h_2\delta h_1}{\left[(h_1)^2 + (h_2)^2\right]^{3/2}} .$$

δh_1 and δh_2 are each the result of variations both in t_1 and in a. Consider first the part of the variation $(h_1\delta h_2 - h_2\delta h_1)$ due to a variation in a.

$$(58)\qquad h_1\frac{\partial h_2}{\partial a} - h_2\frac{\partial h_1}{\partial a} = \left[(k^{11}+ak^{12})b_{11} + (k^{21}+ak^{22})b_{21}\right]\left[k^{12}b_{12} + k^{22}b_{22}\right]$$

$$- \left[(k^{11}+ak^{12})b_{12} + (k^{21}+ak^{22})b_{22}\right]\left[k^{12}b_{11}+k^{22}b_{21}\right]$$

$$= a\left[(k^{12}b_{11}+k^{22}b_{21})(k^{12}b_{12}+k^{22}b_{22})-(k^{12}b_{12}+k^{22}b_{22})(k^{12}b_{11} +k^{22}b_{21})\right] + \left[(k^{11}b_{11}+k^{21}b_{21})(k^{12}b_{12}+k^{22}b_{22}) - (k^{11}b_{12}\right.$$

$$+k^{21}b_{22})(k^{12}b_{11} + k^{22}b_{21})]$$

$$= k^{11}k^{12}(b_{11}b_{12}-b_{12}b_{11}) + k^{21}k^{22}(b_{21}b_{22}-b_{22}b_{21})$$

$$+k^{11}k^{22}(b_{11}b_{22}-b_{12}b_{21}) + k^{21}k^{12}(b_{21}b_{12}-b_{22}b_{11})$$

$$= (k^{11}k^{22}-k^{21}k^{12})(b_{11}b_{22}-b_{12}b_{21})$$

$$= |\mathcal{K}| \cdot |B|,$$

where $|\mathcal{K}|$ denotes the determinant of $\mathcal{K}$ and similarly for $|B|$. Hence the variation due to a variation in a is $|\mathcal{K}|\ |B|$ da.

The part of the variation $(h_1\delta h_2-h_2\delta h_1)$ due to a variation in t_1 is

$$(59)\qquad (h_1\frac{\partial h_2}{\partial t_1} - h_2\frac{\partial h_1}{\partial t_1})\, dt_1 = \left[(k^{11}+ak^{12})b_{11}+(k^{21}+ak^{22})b_{21}\right]$$

$$\cdot\left[(\delta k^{11} + a\delta k^{12})b_{12}+ (\delta k^{21}+a\delta k^{22})b_{22}\right]-$$

$$\left[(k^{11} + ak^{12})b_{12} + (k^{21}+ak^{22})b_{22}\right]\left[(\delta k^{11}\right.$$

$$\left.+a\delta k^{12})b_{11} + (\delta k^{21} + a\delta k^{22})b_{21}\right]$$

$$= |B| \begin{vmatrix} k^{11}+ ak^{12} & \delta k^{11}+ a\delta k^{12} \\ k^{21}+ ak^{22} & \delta k^{21}+ a\delta k^{22} \end{vmatrix}$$

For convenience, we shall refer to this variation as VAR(1) henceforth. Note that each of the δk^{ij} can be calculated, since $\delta\mathcal{K}(t_1,t_1) = (A^T-C^T)$ $\mathcal{K}(t_1,t_1)dt_1$ and since $\delta\dot{\mathcal{K}}= -A^T\delta\mathcal{K}$ on $(0,t_1)$.

It is possible to simplify, within the integral, the terms $\mathcal{K}^T(t_1,t)B$ $\begin{pmatrix}-h_2\\ h_1\end{pmatrix}$, as follows:

$$\mathcal{K}^T(t_1,t)B\begin{pmatrix}-h_2\\ h_1\end{pmatrix}= \begin{pmatrix}k^{11} & k^{21}\\ k^{12} & k^{22}\end{pmatrix}\begin{pmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{pmatrix}\begin{pmatrix}-(k^{11}+ak^{12})b_{12}+(k^{21}+ak^{22})b_{22}\\ (k^{11}+ak^{12})b_{11}+(k^{21}+ak^{22})b_{21}\end{pmatrix}$$

$$= \begin{pmatrix}-a\\ 1\end{pmatrix} |B|\ |\mathcal{K}|.$$

Using the above substitutions, equation (57) simplifies to

(60) $$\mathcal{K}^T(t_1,t_1)dX(t_1) = \mathcal{K}^T(t_1,t_1)\dot{X}(t_{1-}) + \int_0^{t_1} \begin{pmatrix}-a\\ 1\end{pmatrix} |B||\mathcal{K}| \frac{|\mathcal{K}|\ |B|\ da + VAR(1)\ dt_1}{\left[(h_1)^2+(h_2)^2\right]^{3/2}}\ dt.$$

$\mathcal{K}^T$ is a non-singular square matrix, hence $(\mathcal{K}^T)^{-1}$ exists, and

(61) $$dX(t_1) = \dot{X}(t_{1-})\ dt_1 + \left[\mathcal{K}^T(t_1,t_1)\right]^{-1} *$$
$$\int_0^{t_1} \begin{pmatrix}-a\\ 1\end{pmatrix} |B|\,|\mathcal{K}| \frac{|\mathcal{K}|\ |B|\ da + VAR(1)\ dt_1}{\left[(h_1)^2 + (h_2)^2\right]^{3/2}}\ dt,$$

where $dX(t_1) = \begin{pmatrix}dx(t_1)\\ dy(t_1)\end{pmatrix}$. For future reference, let us rewrite $dx(t_1)$ as

(62) $$dx(t_1) = \alpha_{11}dt_1 + \alpha_{12}da,$$

where $\alpha_{11} = \dfrac{\partial\, x(t_1)}{\partial\ t_1}$ and $\alpha_{12} = \dfrac{\partial\, x(t_1)}{\partial\ a}$. Thus we have derived an expression for $dx(t_1)$ in terms of variables which we can calculate.

We want next to derive an explicit expression for $dx(T)$. We can rewrite equation (53) as

(63) $$x(T) = K^{1T}(t_1,t_1)X(t_1) + \int_{t_1}^{T} K^{1T}(t)DF^*dt,$$

whence

(64) $$dx(T) = dK^{1T}(t_1,t_1)X(t_1) + K^{1T}(t_1,t_1)dX(t_1)$$
$$- K^{1T}(t)DF^*(t_{1+})dt_1 + \int_{t_1}^{T} K^{1T}(t)D\delta F^*dt.$$

By the discussion preceding equation (57), $dK^{1T}(t_1,t_1) = -K^{1T}(t_1,t_1)Cdt_1$, and $dK^{1T}(t_1,t_1)X(t_1) = -K^{1T}(t_1,t_1)CX(t_1)dt_1$. If we write the differen-

tial for $dy(t_1)$ corresponding to equation (62) as $dy(t_1) = \alpha_{51}dt_1 + \alpha_{52}da$, the product $K^{1T}(t_1,t_1)dX(t_1)$ becomes

$$(k^{11}\alpha_{11} + k^{12}\alpha_{51})dt_1 + (k^{11}\alpha_{12} + k^{12}\alpha_{52})da.$$

By an argument similar to that preceding equation (60), noting that F^* is dependent only on b in the interval (t_1,T), we get

$$\int_{t_1}^{T} K^{1T}(t)D\delta F^* dt = \int_{t_1}^{T} (-b) \frac{|B|^2 |\mathcal{K}|^2 dt_1 db}{\left[(h_1)^2+(h_2)^2\right]^{3/2}} \tag{65}$$

Combining the above results yields

$$dx(T) = -K^{1T}(t_1,t_1)\left[CX(t_1)+DF^*(t_{1+})\right]dt_1 + (k^{11}\alpha_{11}+k^{12}\alpha_{51})dt_1 + (k^{11}\alpha_{12}+k^{12}\alpha_{62})da + \int_{t_1}^{T}(-b)\frac{|B|^2 |\mathcal{K}|^2 dt\, db.}{\left[(h_1)^2+(h_2)^2\right]^{3/2}} \tag{66}$$

Combining terms gives us an equation corresponding to equation (62), namely

$$dx(T) = \alpha_{21}dt_1 + \alpha_{22}da + \alpha_{23}db, \text{ where } \alpha_{21} = \frac{\partial x(T)}{\partial t_1}, \tag{67}$$

$\alpha_{22} = \dfrac{\partial x(T)}{\partial a}$, and $\alpha_{32} = \dfrac{\partial x(T)}{\partial b}$.

$dy(T)$ can be derived much as we derived $dx(T)$; the equation corresponding to equation (66) is

$$dy(T) = -K^{2T}(t_1,t_1)\left[CX(t_1) + DF^*(t_{1+})\right]dt_1 + (k^{21}\alpha_{11}+k^{22}\alpha_{61})dt_1 + (k^{21}\alpha_{12}+k^{22}\alpha_{62})da + \int_{t_1}^{T}\frac{|D|^2 |\mathcal{K}|^2 dt\, db,}{\left[(h_1)^2+(h_2)^2\right]^{3/2}} \tag{68}$$

which we abbreviate as

$$dy(T) = \alpha_{31}dt_1 + \alpha_{32}da + \alpha_{33}db. \tag{69}$$

We thus have three equations to use to determine corrections in t_1, a, and b which will lead to a maximum x(T), namely

$$(62) \qquad dx(t_1) = \alpha_{11}dt_1 + \alpha_{12}da$$

$$(67) \qquad dx(T) = \alpha_{21}dt_1 + \alpha_{22}da + \alpha_{23}db$$

$$(69) \qquad dy(T) = \alpha_{31}dt_1 + \alpha_{32}da + \alpha_{33}db$$

The numerical procedure used for solving for a maximum x(T) is the following: Choose values for t_1, a, and b. With these values, calculate $x(t_1)$, x(T), and y(T) and the α_{ij}, starting from $X(0) = X_o$. Probably the curve so determined will not be admissible, i.e. $x(t_1)$ will not be x_1 and y(T) will not be y_f. To obtain a curve which will be admissible, solve equations (62) and (69) for two of the variables dt_1, da, or db, setting the third equal to zero. It is possible to solve equations (62) and (69) for dt_1, da, and db only if the rank of the matrix

$$(70) \qquad \begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$$

is two, hence this is a criterion for deciding whether any admissible curves exist. If the rank of (70) is two, replace a, b, and t_1 by a + da, b + db, and $t_1 + dt_1$ and repeat the calculations. Continue this process until either the rank of (70) becomes one or until some admissibility criterion is met. Note: In correcting two variables, it is essential that the value of third variable be chosen close to the value it will have on an extremal; otherwise the correction routine may not converge to an admissible path, even when such a path exists. The admissibility criterion used by the author was $|x_1 - x(t_1)| + |y_f - y(T)| \leq 10^{-5}$; for the matrices used this criterion was normally met in less than ten iter-

ations when the determinants $\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{31} & \alpha_{32} \end{vmatrix}$ and $\begin{vmatrix} \alpha_{11} & 0 \\ \alpha_{31} & \alpha_{33} \end{vmatrix}$ went to zero, this was also obvious by the tenth iteration.

When admissibility had been achieved, equations (62), (67), and (69) took on the form

$$(71) \quad 0 = \alpha_{11}dt_1 + \alpha_{12}da$$

$$(72) \quad dx(T) = \alpha_{21}dt_1 + \alpha_{22}da + \alpha_{23}db$$

$$(73) \quad 0 = \alpha_{31}dt_1 + \alpha_{32}da + \alpha_{33}db.$$

These equations were solved for dx(T) unless the rank of

$\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$ was equal to the rank of $\begin{pmatrix} \alpha_{11} & \alpha_{12} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}$. If the rank of the latter matrix was two, this condition was that the determinant of the first matrix must be zero. We therefore want some way to determine a new set of corrections to drive the determinant toward zero. The method adopted was to choose dt_1, da, and db proportional to the components of grad $x(t_1)$ X grad y(T), where "X" indicates the vector cross-product and where the proportionality constant is some fixed number ε multiplied by the determinant $|\alpha_{ij}|$. The old values for t_1,a, and b were replaced by the corrected values and a new admissible curve was found using the method outlined above. For this curve the determinant $|\alpha_{ij}|$ and the corrections da, db, and dt_1 were again calculated together with x(T). This process of finding an admissible curve, then calculating a set of corrections to increase x(T), then finding a new admissible curve, then finding a new set of corrections to increase x(T) was continued until the determinant $|\alpha_{ij}|$ changed sign. When the determinant changed sign, however, the above procedure no longer worked - the corrections became

larger rather than smaller, and the values of x(T) got smaller instead of larger. A procedure that worked when the determinant changed sign was the following: A new proportionality constant was chosen equal to one fifth of the sum of the positive and negative determinants; a new set of corrections da, db, and dt_1 was then calculated. The admissible curve calculated after finding this new set of corrections invariably yielded a larger x(T) than had been obtained before. This method of successive approximations seems to be a nearly foolproof method for determining the maximum x(T). The procedure was stopped when either the absolute value of the determinant was less than 10^{-5} or it became obvious that the convergence was too slow for the amount of time available on the computer. If convergence was too slow, it could usually be improved by increasing the number ε referred to above.

Another successive approximation procedure used was to compare the x(T) resulting from replacing t_1,a, and b by their corrected values with the previous x(T). If the new x(T) were smaller, ε was reduced by a factor of ten and a new set of corrections and hence a new x(T) computed. This process was continued until either the sum of the absolute values of the corrections was less than 10^{-5} or the new x(T) was larger than the old x(T), in which case the new x(T) became the value compared with. This process gives a monotonically increasing sequence of values of x(T); the speed of computation was comparable to that of the other method.

It was necessary to insure that the corrections from the routine were not so large as to make the linearity approximations in the correction integrals invalid. The author used the following procedure: In the correction to admissibility, if either da or db exceeded .3 in absolute value, or if dt_1 exceeded .5 in absolute value, he divided all the

corrections in half. For the problems worked, this criterion was adequate although in several cases it slowed convergence quite a bit. (For example, where the initial value given for a was 1. and where the final value on the admissible curve was 12.8.) Once admissibility was achieved, it was necessary to use some linearity criterion on the second set of corrections. The procedure was to divide the corrections in half if the sum of the absolute values of the corrections exceeded .5.

With the above correction procedures, computations were made on several sets of matrices with various boundary conditions. The only case where it was possible to verify the results by comparison with a physical situation where A and C were set equal to zero and where B and D were 2 x 2 scalar matrices. This set of matrices gives the differential equation $\dot{X} = \begin{cases} aF, & 0 < t < t_1 \\ cF, & t_1 < t < T \end{cases}$. The constraint on F was that $(f_1)^2 + (f_2)^2 = 1$; f_1 and f_2 were consequently chosen as $\cos\alpha$ and $\sin\alpha$, for $0 < t < t_1$, and as $\cos\beta$ and $\sin\beta$, for $t_1 < t < T$. Solutions to the differential equation as set up describe the path taken by a light ray going from one isotropic medium into another, with α and β being the angles between the light ray and the normals to the plane of discontinuity. As such, these solutions should obey Snell's Law, namely that the ratio of the velocities on opposite sides of the discontinuity plae should be the same as the ratio of the sines of the angles the ray makes with the normal to that plane. The cases tried were for $X_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $x_1 = 1.0$, $y_f = 5.0$, a = 1.0 and b = 2.0 and, in the second case, a = 2.0 and b = 1.0. For the first case, the ratio between the sines should be .5. α was found to be approximately .403378 radians and β approximately .902758 radians. The ratio of the sines of these angles was .500, which verified the accuracy of the routine. For a = 2.0 and b = 1.0, the angles were reversed, and again the

accuracy of the routine was verified.

Another case tried was where $A = \begin{pmatrix} 1.0 & .1 \\ 1.2 & 1.0 \end{pmatrix}$, $B = \begin{pmatrix} 1.0 & .2 \\ .0 & 1.1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, and $D = \begin{pmatrix} 2.0 & 0. \\ 0. & 2.0 \end{pmatrix}$, with $x_1 = 1.0$, $T = 4.0$, and where initial values for t_1, $\tan\alpha$, and $\tan\beta$ were .73, 4.0, and .7, respectively. The routine converged to $t_1 = .8935$, $\tan\alpha = 1.4925$, and $\tan\beta = .4967$, with the maximum $x(T)$ being 6.4622. This convergence took fifteen minutes and fifty-one seconds on a CDC 1604 computer. The author felt that convergence could be improved if better initial estimates were made of t_1, $\tan\alpha$, and $\tan\beta$.

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APPENDIX I

NOTES

1. FLOW CHART PAGE NUMBERS ARE DESIGNATED BY THE LETTER F, e.g., F6 IS PAGE 6 OF THE FLOW CHART.
2. CONNECTORS ARE REFERENCED BY PAGE NUMBERS, e.g., (3) F2 DESIGNATES CONNECTOR 3 IS ON PAGE F2.

Enter

These constants are given:

x(0), y(0), XMID, YFINAL, MATRICES A, B, C, and D, EPSFAC, X11, ATC12, ATD11, TONE, CGUESS, DGUESS, CAPT, ERROR, EPS1

(1)

SET STEP SIZE

TSTEP(1) = (CAPT - TONE)/50

TSTEP(2) = TONE/50

INTEGRATION FROM t = CAPT to t = TONE

In: $K(\text{CAPT}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Integrate: $\dot{K} = -C^T K$

Out: $K(t_1)$

Calculate: $\delta K(t_1) = (A^T - C^T) K(t_1)$

INTEGRATION FROM t = TONE to t = 0

In: $K(t_1)$, $\delta K(t_1)$

Integrate: $\dot{K} = -A^T K$, $\delta\dot{K} = -A^T \delta K$

Out: $\dot{K}(0)$, $\delta K(0)$

INTEGRATION FROM t = 0 to t = TONE

In: $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$, $K(0)$, $\delta K(0)$. Note: $K = \left(K^1 : K^2\right)$; $\delta K = \left(\delta K^1 : \delta K^2\right)$

Integrate: $\dot{K} = -A^T K$, $\delta\dot{K} = -A^T \delta K$

Calculate: $\left(\text{ACON}(1) \;\; \text{ACON}(2)\right) = \left(K^{1T} + \text{CGUESS} \times K^{2T}\right) B$

$$\text{SRT} = \left(\left(\text{ACON}(1)\right)^2 + \left(\text{ACON}(2)\right)^2\right)^{1/2}$$

$$F = \begin{pmatrix} \text{ACON}(1) \\ \text{ACON}(2) \end{pmatrix} \div \text{SRT}$$

Integrate: $\dot{X} = AX + BF$

$$d\left(\text{CORR}(1)\right) = \left(-\text{CGUESS} \times |B|^2 \times |K|^2 \div (\text{SRT})^3\right) dt$$

$$d\left(\text{CORR}(2)\right) = \left(|B|^2 \times |K|^2 \div (\text{SRT})^3\right) dt$$

Calculate: $$\text{DINTFAC} = |B| \begin{vmatrix} K^{11} + \text{CGUESS} \times K^{12} & \delta K^{11} + \text{CGUESS} \times \delta K^{12} \\ K^{21} + \text{CGUESS} \times K^{22} & \delta K^{21} + \text{CGUESS} \times \delta K^{22} \end{vmatrix}$$

Integrate: $$d\left(\text{CORR}(3)\right) = -\text{CGUESS} \times |B|^2 \times |K| \times \text{DINTFAC} \div (\text{SRT})^3\, dt$$

$$d\left(\text{CORR}(4)\right) = |B|^2 \times |K| \times \text{DINTFAC} \div (\text{SRT})^3\, dt$$

Out: $\begin{pmatrix} x(t_1) \\ y(t_1) \end{pmatrix}$, $K(t_{1-})$, $\dot{X}(t_{1-}) = \begin{pmatrix} \text{XDOT1} \\ \text{YDOT1} \end{pmatrix}$, CORR(1), CORR(2), CORR(3), CORR(4)

(2) F1

(2)

Set XINMID = $x(t_1)$

Calculate CKI = $\left(K^T(t_1)\right)^{-1}$

CHANGES IN $X(t_1)$ AND Y DUE TO CHANGES IN CGUESS

XONE(2) = (1 0) CKI $\begin{pmatrix} \text{CORR}(1) \\ \text{CORR}(2) \end{pmatrix}$

YONE(2) = CORR(2)

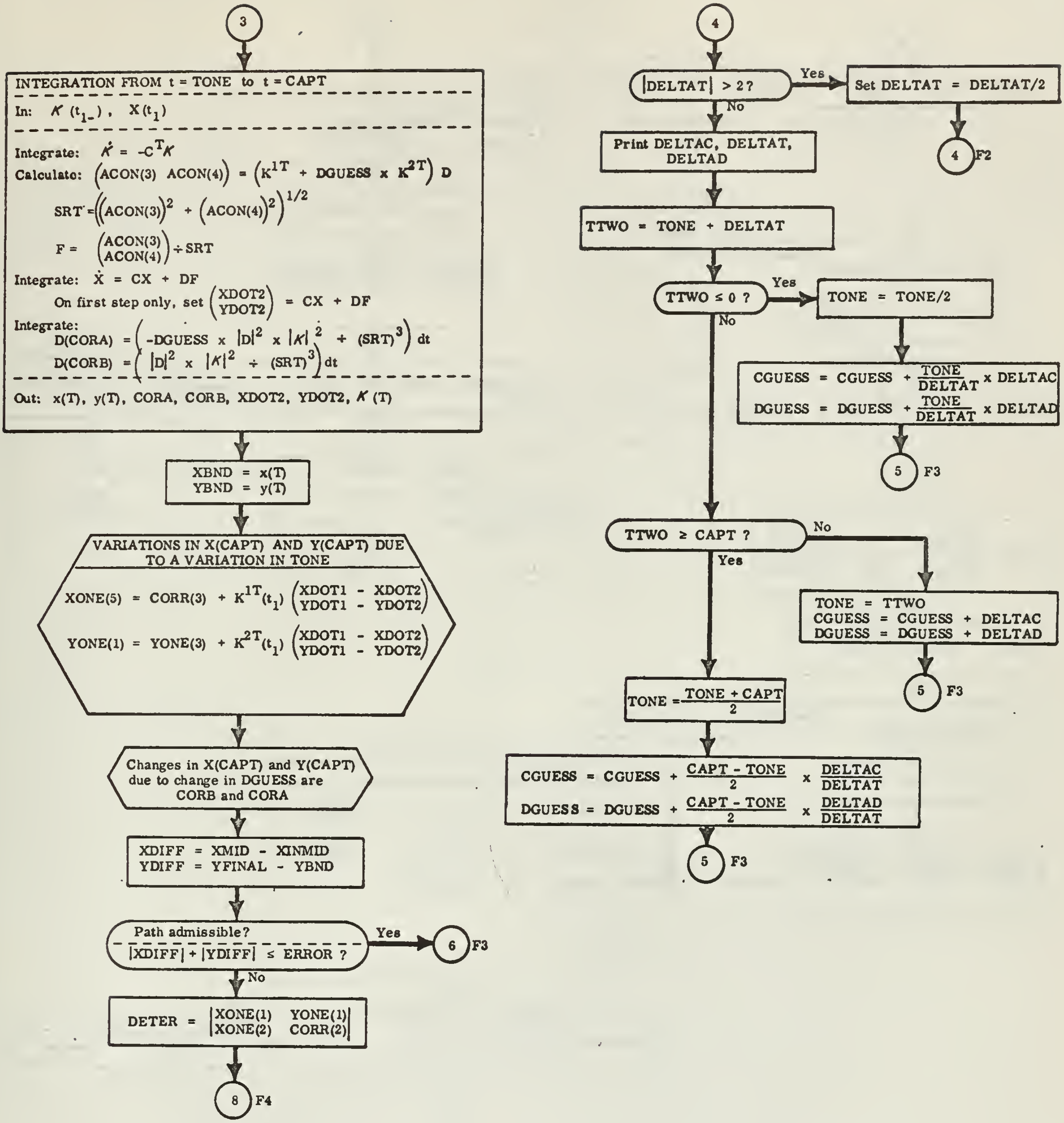
VARIATION IN X(TONE) DUE TO A VARIATION IN TONE

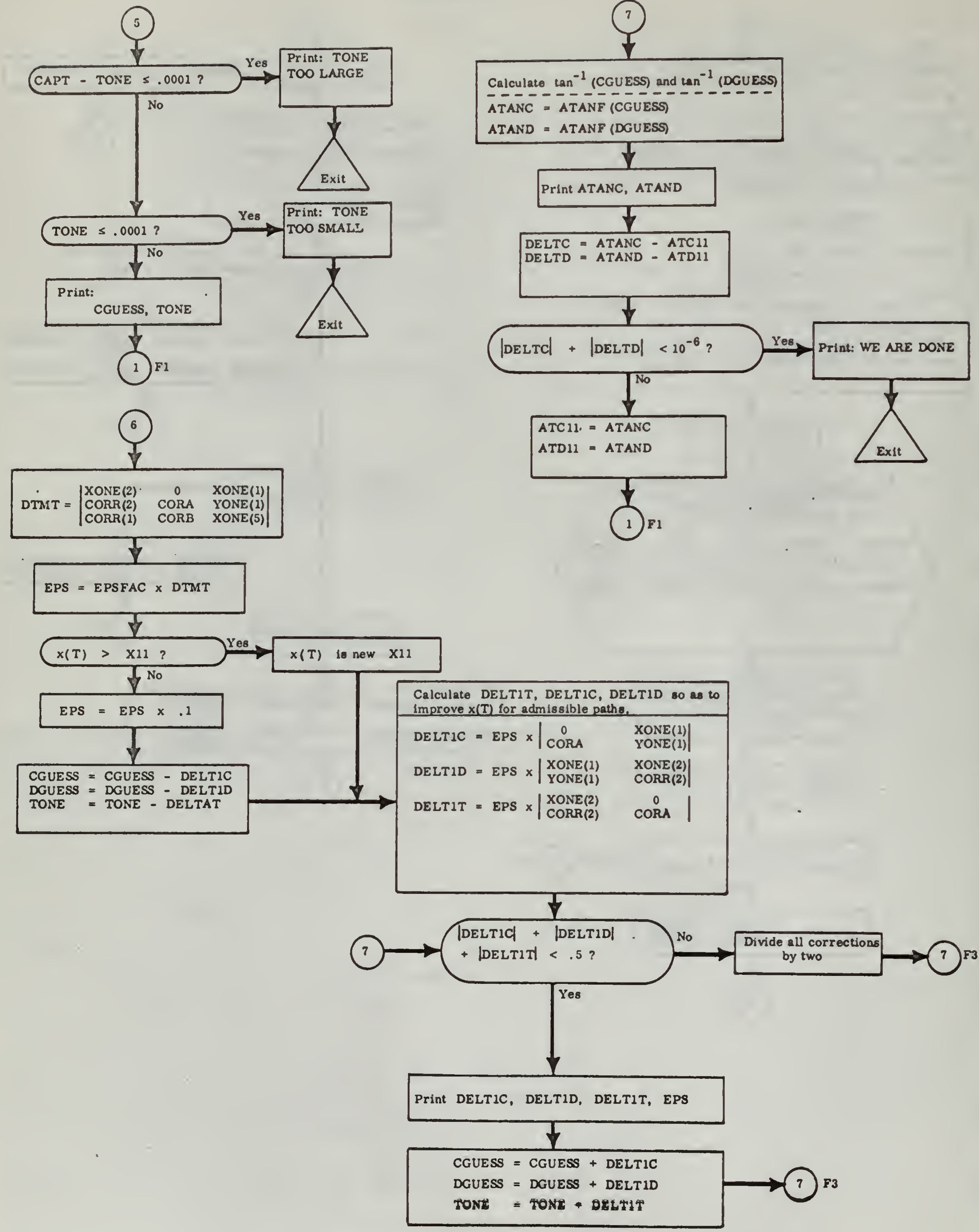
XONE(1) = (1 0) CKI $\begin{pmatrix} \text{CORR}(3) \\ \text{CORR}(4) \end{pmatrix}$ + XDOT1

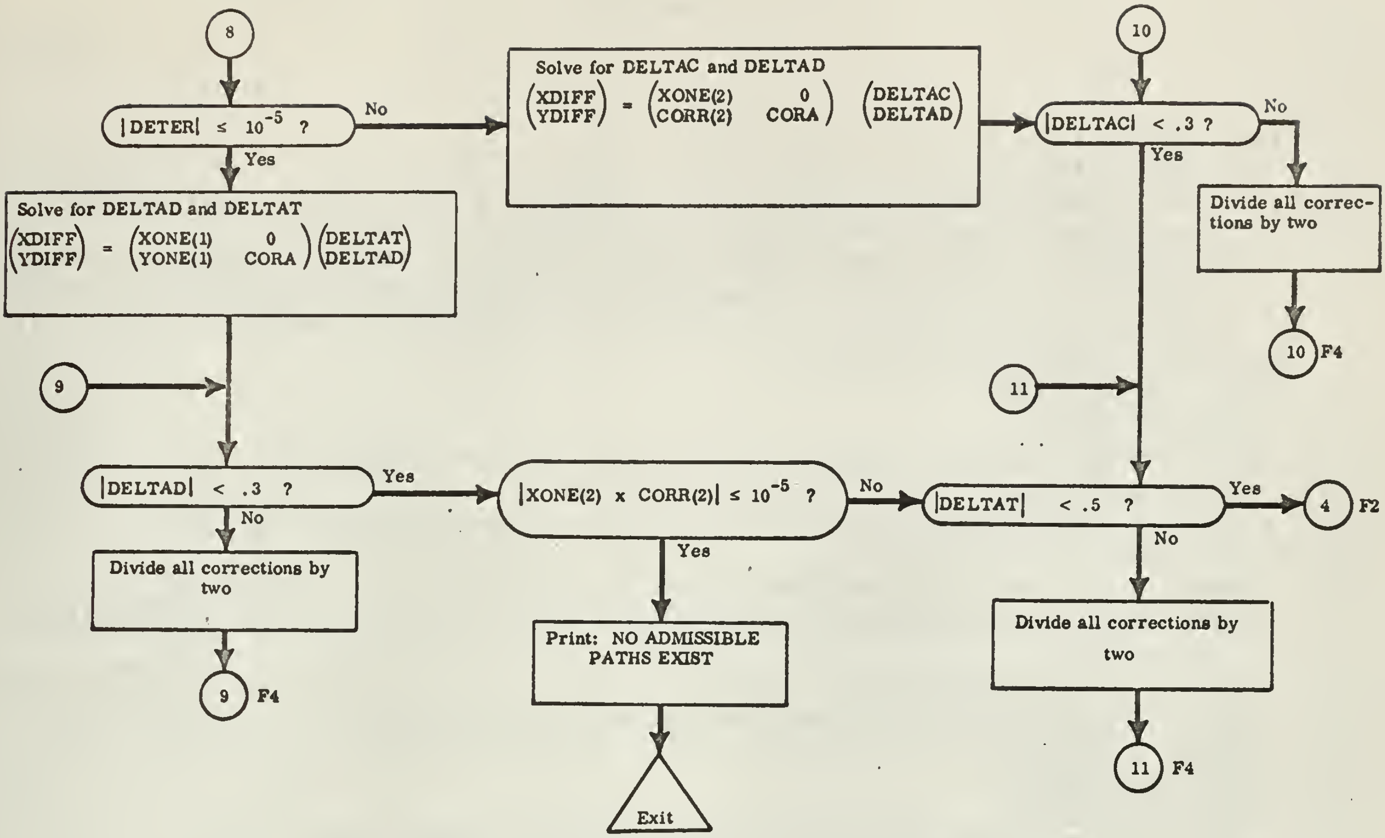
PARTIAL VARIATION IN Y DUE TO A VARIATION IN TONE

YONE(3) = CORR(4)

(3) F2


3
INTEGRATION FROM t = TONE to t = CAPT
In: K (t1-), X(t1)
Integrate: K̇ = -C^T K
Calculate: (ACON(3) ACON(4)) = (K^1T + DGUESS x K^2T) D
SRT = ((ACON(3))^2 + (ACON(4))^2)^1/2
F = (ACON(3) ACON(4)) ÷ SRT
Integrate: Ẋ = CX + DF
On first step only, set (XDOT2 YDOT2) = CX + DF
Integrate:
D(CORA) = (-DGUESS x |D|^2 x |K|^2 + (SRT)^3) dt
D(CORB) = (|D|^2 x |K|^2 ÷ (SRT)^3) dt
Out: x(T), y(T), CORA, CORB, XDOT2, YDOT2, K (T)
XBND = x(T)
YBND = y(T)
VARIATIONS IN X(CAPT) AND Y(CAPT) DUE TO A VARIATION IN TONE
XONE(5) = CORR(3) + K^1T(t1) (XDOT1 - XDOT2 / YDOT1 - YDOT2)
YONE(1) = YONE(3) + K^2T(t1) (XDOT1 - XDOT2 / YDOT1 - YDOT2)
Changes in X(CAPT) and Y(CAPT) due to change in DGUESS are CORB and CORA
XDIFF = XMID - XINMID
YDIFF = YFINAL - YBND
Path admissible?
|XDIFF| + |YDIFF| ≤ ERROR ?
Yes
6 F3
No
DETER = |XONE(1) YONE(1) / XONE(2) CORR(2)|
8 F4
4
|DELTAT| > 2?
Yes
Set DELTAT = DELTAT/2
4 F2
No
Print DELTAC, DELTAT, DELTAD
TTWO = TONE + DELTAT
TTWO ≤ 0 ?
Yes
TONE = TONE/2
CGUESS = CGUESS + TONE/DELTAT x DELTAC
DGUESS = DGUESS + TONE/DELTAT x DELTAD
5 F3
No
TTWO ≥ CAPT ?
No
TONE = TTWO
CGUESS = CGUESS + DELTAC
DGUESS = DGUESS + DELTAD
5 F3
Yes
TONE = (TONE + CAPT)/2
CGUESS = CGUESS + (CAPT - TONE)/2 x DELTAC/DELTAT
DGUESS = DGUESS + (CAPT - TONE)/2 x DELTAD/DELTAT
5 F3


5
CAPT - TONE ≤ .0001 ?
Yes
Print: TONE TOO LARGE
Exit
No
TONE ≤ .0001 ?
Yes
Print: TONE TOO SMALL
Exit
No
Print: CGUESS, TONE
1 F1
6
DTMT = |XONE(2) 0 XONE(1) / CORR(2) CORA YONE(1) / CORR(1) CORB XONE(5)|
EPS = EPSFAC x DTMT
x(T) > X11 ?
Yes
x(T) is new X11
No
EPS = EPS x .1
CGUESS = CGUESS - DELT1C
DGUESS = DGUESS - DELT1D
TONE = TONE - DELTAT
Calculate DELT1T, DELT1C, DELT1D so as to improve x(T) for admissible paths.
DELT1C = EPS x |0 XONE(1) / CORA YONE(1)|
DELT1D = EPS x |XONE(1) XONE(2) / YONE(1) CORR(2)|
DELT1T = EPS x |XONE(2) 0 / CORR(2) CORA|
7
|DELT1C| + |DELT1D| + |DELT1T| < .5 ?
No
Divide all corrections by two
7 F3
Yes
Print DELT1C, DELT1D, DELT1T, EPS
CGUESS = CGUESS + DELT1C
DGUESS = DGUESS + DELT1D
TONE = TONE + DELT1T
7 F3
7
Calculate tan⁻¹ (CGUESS) and tan⁻¹ (DGUESS)
ATANC = ATANF (CGUESS)
ATAND = ATANF (DGUESS)
Print ATANC, ATAND
DELTC = ATANC - ATC11
DELTD = ATAND - ATD11
|DELTC| + |DELTD| < 10⁻⁶ ?
Yes
Print: WE ARE DONE
Exit
No
ATC11 = ATANC
ATD11 = ATAND
1 F1


8
|DETER| ≤ 10^-5 ?
No
Yes
Solve for DELTAD and DELTAT
(XDIFF YDIFF) = (XONE(1) 0 YONE(1) CORA) (DELTAT DELTAD)
9
|DELTAD| < .3 ?
Yes
No
Divide all corrections by two
9 F4
Solve for DELTAC and DELTAD
(XDIFF YDIFF) = (XONE(2) 0 CORR(2) CORA) (DELTAC DELTAD)
|XONE(2) x CORR(2)| ≤ 10^-5 ?
No
Yes
Print: NO ADMISSIBLE PATHS EXIST
Exit
10
|DELTAC| < .3 ?
No
Yes
Divide all corrections by two
10 F4
11
|DELTAT| < .5 ?
Yes
4 F2
No
Divide all corrections by two
11 F4

APPENDIX II

```
      PROGRAM DISCON
C     THIS PROGRAM COMPUTES MAX X(CAPT) FOR A GIVEN Y(CAPT) AND A GIVEN
C         X(T1),WHERE XDOT=AX+BF,FOR T BETWEEN 0 AND T1,AND CX+DF,FOR T
C         BETWEEN T1 AND CAPT. CAPT IS GIVEN,BUT T1 IS NOT.
      DIMENSION A(2,2),B(2,2),C(2,2),D(2,2),BK(2,2),F(4),TSTEP(2),E(2,2,
     1  2),ELV(14),DE(16),         AAK(2,2),DP(16),ACON(4),AKTB(2,2),CORR
     2   (4),CKI(2,2),FLES(2),XONE(5),YONE(4),BKA(2,2),FMOR(2),DELK(2,2)
     3,DACON(4),DFE(4),AK(5,16)
      READ 1503,EPSFAC
 1503 FORMAT(1F20.10)
      READ 501,((A(I,J),J=1,2),I=1,2),((B(I,J),J=1,2),I=1,2)
  501 FORMAT (8F10.1)
      READ 501,((C(I,J),J=1,2),I=1,2),((D(I,J),J=1,2),I=1,2)
      PRINT 551
  551 FORMAT (14X,1HA,29X,1HB,29X,1HC,29X,1HD)
      PRINT 552,((A(I,J),J=1,2),(B(I,J),J=1,2),(C(I,J),J=1,2),(D(I,J),J=
     +1,2),I=1,2)
  552 FORMAT(4(8X,F4.1,6X,F4.1,8X))
      READ 502,TONE,X0,Y0,YFINAL,XMID,CGUESS,ERROR,CAPT
  502 FORMAT (8F10.8)
      READ 503,DGUESS
  503 FORMAT(F10.8)
      PRINT 553,TONE,X0,Y0,YFINAL,XMID,CGUESS,ERROR,CAPT
  553 FORMAT(4X,12HTONE GUESS= ,F5.2, 9H(X0,Y0)=(,F4.1,1H,,F4.1, 9H)YFIN
     +AL= ,F4.1,7H XMID= ,F5.2,9H CGUESS= ,F4.1,8H ERROR= ,F7.6,6H CAPT=
     +,F5.2)
C     INTEGRATE ADJOINT BACKWARDS TO GET K(0) AND KINVERSE AT T1.
      F(1)=0.
      F(2)=0.5
      F(3)=0.5
      F(4)=1.
      ATC11=5.
      ATD11=5.
```

```
      X11=1.
      EPS2=-.01
    1 TSTEP(2)=TONE/50.
      DO 1003 I=1,3
      YONE(I)=0.
 1003 XONE(I)=0.
      TSTEP(1)=(CAPT-TONE)/50.
      TSTEP(1)=-TSTEP(1)
      TSTEP(2)=-TSTEP(2)
      BK(1,1)=1.
      BK(2,1)=0.
      BK(1,2)=0.
      BK(2,2)=1.
      T=CAPT
      DO 101 I=1,2
      DO 101 J=1,2
      E(2,I,J)=A(I,J)
  101 E(1,I,J)=C(I,J)
      ELV(1)=BK(1,1)
      ELV(2)=BK(1,2)
      ELV(3)=BK(2,1)
      ELV(4)=BK(2,2)
      DO 1004 IMA=5,8
 1004 ELV(IMA)=0.
      IB=1
 1001 DO 102 IE=1,50
      DO 103 IC=1,4
      DO 104 ID=1,4
  104 DE(ID)=ELV(ID)+F(IC)*AK(IC-1,ID)
      DP(1)=-E(IB,1,1)*DE(1)-E(IB,2,1)*DE(3)
      DP(3)=-E(IB,1,2)*DE(1)-E(IB,2,2)*DE(3)
      DP(2)=-E(IB,1,1)*DE(2)-E(IB,2,1)*DE(4)
      DP(4)=-E(IB,1,2)*DE(2)-E(IB,2,2)*DE(4)
      DO 103 ID=1,4
  103 AK(IC,ID)=TSTEP(IB)*DP(ID)
      DO 105 ID=1,4
```

```
 105 ELV(ID)=ELV(ID)+(AK(1,ID)+2.*AK(2,ID)+2.*AK(3,ID)+AK(4,ID))/6.
 102 T=T+TSTEP(IB)
     BK(1,1)=ELV(1)
     BK(1,2)=ELV(2)
     BK(2,1)=ELV(3)
     BK(2,2)=ELV(4)
     IB=2
1002 ELV(5)=(A(1,1)-C(1,1))*BK(1,1)+(A(2,1)-C(2,1))*BK(2,1)
     ELV(6)=(A(1,1)-C(1,1))*BK(1,2)+(A(2,1)-C(2,1))*BK(2,2)
     ELV(7)=(A(1,2)-C(1,2))*BK(1,1)+(A(2,2)-C(2,2))*BK(2,1)
     ELV(8)=(A(1,2)-C(1,2))*BK(1,2)+(A(2,2)-C(2,2))*BK(2,2)
     DO 112 IA=1,50
     DO 903 IC=1,4
     DO 904 ID=1,8
 904 DE(ID)=ELV(ID)+F(IC)*AK(IC-1,ID)
     DP(1)=-E(IB,1,1)*DE(1)-E(IB,2,1)*DE(3)
     DP(3)=-E(IB,1,2)*DE(1)-E(IB,2,2)*DE(3)
     DP(2)=-E(IB,1,1)*DE(2)-E(IB,2,1)*DE(4)
     DP(4)=-E(IB,1,2)*DE(2)-E(IB,2,2)*DE(4)
     DP(5)=-E(IB,1,1)*DE(5)-E(IB,2,1)*DE(7)
     DP(7)=-E(IB,1,2)*DE(5)-E(IB,2,2)*DE(7)
     DP(6)=-E(IB,1,1)*DE(6)-E(IB,2,1)*DE(8)
     DP(8)=-E(IB,1,2)*DE(6)-E(IB,2,2)*DE(8)
     DO 903 ID=1,8
 903 AK(IC,ID)=TSTEP(IB)*DP(ID)
     DO 905 ID=1,8
 905 ELV(ID)=ELV(ID)+(AK(1,ID)+2.*AK(2,ID)+2.*AK(3,ID)+AK(4,ID))/6.
 112 T=T+TSTEP(IB)
     BK(1,1)=ELV(1)
     BK(1,2)=ELV(2)
     BK(2,1)=ELV(3)
     BK(2,2)=ELV(4)
     DELK(1,1)=ELV(5)
     DELK(1,2)=ELV(6)
     DELK(2,1)=ELV(7)
     DELK(2,2)=ELV(8)
```

```
      DELK IS THE MATRIX OF DELK S AT T=0
      TSTEP(1)=-TSTEP(1)
      TSTEP(2)=-TSTEP(2)
      THE K MATRIX, AS IS, IS AT T=0.
      THE NEXT INTEGRATION IS FROM T=0 TO T=TONE
      ELV(1)=BK(1,1)
      ELV(2)=BK(1,2)
      ELV(3)=BK(2,1)
      ELV(4)=BK(2,2)
      ELV(5)=0.
      ELV(6)=0.
      ELV(7)=X0
      ELV(8)=Y0
      ELV(9)=DELK(1,1)
      ELV(10)=DELK(1,2)
      ELV(11)=DELK(2,1)
      ELV(12)=DELK(2,2)
      ELV(13)=0.
      ELV(14)=0.
      DO 202 IA=1,50
      DO 203 IC=1,4
      DO 204 ID=1,14
  204 DE(ID)=ELV(ID)+F(IC)*AK(IC-1,ID)
      DP(1)=-A(1,1)*DE(1)-A(2,1)*DE(3)
      DP(3)=-A(1,2)*DE(1)-A(2,2)*DE(3)
      DP(2)=-A(1,1)*DE(2)-A(2,1)*DE(4)
      DP(4)=-A(1,2)*DE(2)-A(2,2)*DE(4)
      ACON(1)= DE(1)*B(1,1)+DE(3)*B(2,1)+CGUESS*(DE(2)*B(1,1)+DE(4)*B(2,
     +1))
      ACON(2)=DE(1)*B(1,2)+DE(3)*B(2,2)+CGUESS*(DE(2)*B(1,2)+DE(4)*B(2,
     +2))
      SRT=SQRTF(ACON(1)*ACON(1)+ACON(2)*ACON(2))
      AKDET=DE(1)*DE(4)-DE(2)*DE(3)
      BDET=B(1,1)*B(2,2)-B(2,1)*B(1,2)
      DP(5)=-CGUESS*AKDET*AKDET*BDET*BDET/SRT**3
      DP(6)=        AKDET*AKDET*BDET*BDET/SRT**3
```

```
      DP(7)=A(1,1)*DE(7)+A(1,2)*DE(8)+(B(1,1)*ACON(1)+B(1,2)*ACON(2))/SR
     +T
      DP(8)=A(2,1)*DE(7)+A(2,2)*DE(8)+(B(2,1)*ACON(1)+B(2,2)*ACON(2))/SR
     +T
      DP(9)=-A(1,1)*DE(9)-A(2,1)*DE(11)
      DP(11)=-A(1,2)*DE(9)-A(2,2)*DE(11)
      DP(10)=-A(1,1)*DE(10)-A(2,1)*DE(12)
      DP(12)=-A(1,2)*DE(10)-A(2,2)*DE(12)
      DINTFAC=CGUESS*(DE(1)*DE(12)-DE(4)*DE(9)+DE(2)*DE(11)-DE(3)*DE(10)
     1      +CGUESS*(DE(2)*DE(12)-DE(4)*DE(10)))-DE(1)*DE(11)+DE(3)*DE(9)
      DP(13)=-CGUESS*BDET*BDET*AKDET*DINTFAC/SRT**3
      DP(14)=        BDET*BDET*AKDET*DINTFAC/SRT**3
      DO 203 ID=1,14
  203 AK(IC,ID)=TSTEP(2)*DP(ID)
      DO 206 ID=1,14
  206 ELV(ID)=ELV(ID)+(AK(1,ID)+2.*AK(2,ID)+2.*AK(3,ID)+AK(4,ID))/6.
      BK(1,1)=ELV(1)
      BK(1,2)=ELV(2)
      BK(2,1)=ELV(3)
      BK(2,2)=ELV(4)
      CORR(1)=ELV(5)
      CORR(2)=ELV(6)
      XMOD=ELV(7)
      YMOD=ELV(8)
      DELK(1,1)=ELV(9)
      DELK(1,2)=ELV(10)
      DELK(2,1)=ELV(11)
      DELK(2,2)=ELV(12)
      CORR(3)=ELV(13)
      CORR(4)=ELV(14)
  202 T=T+TSTEP(2)
C     CORR 3 AND 4 GIVE THE VARIATIONS OF X AND Y AT T1 DUE TO DT1
C     GET K INVERSE AT TONE AND F MINUS
      AKDET=BK(1,1)*BK(2,2)-BK(1,2)*BK(2,1)
      XDOT1=DP(7)
      YDOT1=DP(8)
```

```
      CKI(1,1)=BK(2,2)/AKDET
      CKI(1,2)=-BK(2,1)/AKDET
      CKI(2,1)=-BK(1,2)/AKDET
      CKI(2,2)=BK(1,1)/AKDET
      FLES(1)=ACON(1)/SRT
      FLES(2)=ACON(2)/SRT
      YONE(3)=CORR(4)
      XONE(3)=CKI(1,1)*CORR(3)+CKI(1,2)*CORR(4)
      XONE(1)=XONE(3)+XDOT1
      XONE(2)=      (CKI(1,1)*CORR(1)+CKI(1,2)*CORR(2))
      YONE(2)=CORR(2)
      XINMID=XMOD
      PRINT 559,XINMID
  559 FORMAT(20H 559  X AT TONE IS   ,F20.10)
      INTEGRATE FROM T1 TO CAPT
      ELV(1)=BK(1,1)
      ELV(2)=BK(1,2)
      ELV(3)=BK(2,1)
      ELV(4)=BK(2,2)
      ELV(5)=XMOD
      ELV(6)=YMOD
      ELV(7)=0.
      ELV(8)=0.
      DO 302 IA=1,50
      DO 303 IC=1,4
      DO 304 ID=1,8
  304 DE(ID)=ELV(ID)+F(IC)*AK(IC-1,ID)
      DP(1)=-C(1,1)*DE(1)-C(2,1)*DE(3)
      DP(3)=-C(1,2)*DE(1)-C(2,2)*DE(3)
      DP(2)=-C(1,1)*DE(2)-C(2,1)*DE(4)
      DP(4)=-C(1,2)*DE(2)-C(2,2)*DE(4)
      ACON(3)=DE(1)*D(1,1)+DE(3)*D(2,1)+DGUESS*(DE(2)*D(1,1)+DE(4)*D(2,1
     +))
      ACON(4)=DE(1)*D(1,2)+DE(3)*D(2,2)+DGUESS*(DE(2)*D(1,2)+DE(4)*D(2,2
     +))
      SRT=SQRTF(ACON(3)*ACON(3)+ACON(4)*ACON(4))
```

```
      CKDET=DE(1)*DE(4)-DE(2)*DE(3)
      DDET=D(1,1)*D(2,2)-D(1,2)*D(2,1)
      DP(5)=C(1,1)*DE(5)+C(1,2)*DE(6)+(D(1,1)*ACON(3)+D(1,2)*ACON(4))/SR
     +T
      DP(6)=C(2,1)*DE(5)+C(2,2)*DE(6)+(D(2,1)*ACON(3)+D(2,2)*ACON(4))/SR
     +T
      DP(7)=        DDET*DDET*CKDET*CKDET/SRT**3
      DP(8)=-DGUESS*DDET*DDET*CKDET*CKDET/SRT**3
      DO 303 ID=1,8
  303 AK(IC,ID)=TSTEP(1)*DP(ID)
      IF(IA-1)801,801,802
  801 CONTINUE
      XDOT2=DP(5)
      YDOT2=DP(6)
  802 CONTINUE
      DO 306 ID=1,8
  306 ELV(ID)=ELV(ID)+(AK(1,ID)+2.*AK(2,ID)+2.*AK(3,ID)+AK(4,ID))/6.
  302 T=T+TSTEP(1)
      BKA(1,1)=ELV(1)
      BKA(1,2)=ELV(2)
      BKA(2,1)=ELV(3)
      BKA(2,2)=ELV(4)
      XMOD=ELV(5)
      YMOD=ELV(6)
      CORA=ELV(7)
      CORB=ELV(8)
C     THIS COMPLETES THE INTEGRATION
      XONE(5)=CORR(3)+BK(1,1)*(XDOT1-XDOT2)+BK(2,1)*(YDOT1-YDOT2)
      YONE(1)=CORR(4)+BK(1,2)*(XDOT1-XDOT2)+BK(2,2)*(YDOT1-YDOT2)
      XBND=XMOD
      YBND=YMOD
      PRINT 558,XBND,YBND
  558 FORMAT(4H 558,9H  X(T)=  ,F20.10,9H  Y(T)=  ,F20.10)
      XDIFF=XMID-XINMID
      YDIFF=YFINAL-YBND
      IF(ABSF(XDIFF)+ABSF(YDIFF)-ERROR)998,998,6001
```

```
6001 DETER=XONE(1)*CORR(2)-XONE(2)*YONE(1)
     IF(ABSF(DETER)-1.E-5)6002,6002,6011
6011 DELTAC=XDIFF/XONE(2)
     DELTAD=(YDIFF-CORR(2)*DELTAC)/CORA
     DELTAT=0.
6670 IF(ABSF(DELTAC)-.3)6671,6672,6672
6672 DELTAC=.5*DELTAC
     DELTAD=.5*DELTAD
     DELTAT=.5*DELTAT
     GO TO 6670
6671 GO TO 9001
6002 DELTAT=XDIFF/XONE(1)
     DELTAD=(YDIFF-YONE(1)*XDIFF/XONE(1))/CORA
6673 IF(ABSF(DELTAD)-.3)6674,6675,6675
6675 DELTAD=.5*DELTAD
     DELTAC=.5*DELTAC
     DELTAT=.5*DELTAT
     GO TO 6673
6674 CONTINUE
     DELTAC=0.
     DETER=XONE(1)*CORA
     IF(ABSF(DETER)-1.E-5)9198,9198,9001
9001 IF(ABSF(DELTAT)-.5)990,9002,9002
9002 DELTAT=.5*DELTAT
     DELTAC=.5*DELTAC
     DELTAD=.5*DELTAD
     GO TO 9001
 990 PRINT 562,DELTAC,DELTAT,DELTAD
 562 FORMAT(12H  DELTAC =  ,F20.10,14H  DELTA T1 =  ,F20.10,12H  DELTAD
    + =  ,F20.10)
     TTWO=TONE+DELTAT
     IF(TTWO)996,996,309
 996 TONE=TONE*.5
     CGUESS=CGUESS+(TONE/DELTAT*DELTAC)
     DGUESS=DGUESS+(TONE/DELTAT*DELTAD)
     GO TO 310
```

```
 309 IF(TTWO-CAPT)311,995,995
 995 TONE=(TONE+CAPT)*.5
     CGUESS=CGUESS+(CAPT-TONE)*.5/DELTAT*DELTAC
     DGUESS=DGUESS+(CAPT-TONE)*.5/DELTAT*DELTAD
     GO TO 310
 311 TONE=TTWO
     CGUESS=CGUESS+DELTAC
     DGUESS=DGUESS+DELTAD
 310 CONTINUE
     IF(CAPT-TONE-1.E-6)9771,9771,9772
9771 PRINT 9781
9781 FORMAT(5H 9781,16H  TONE TOO LARGE)
     GO TO 9991
9772 IF(TONE-1.E-6)9773,9773,9774
9773 PRINT 9782
9782 FORMAT(5H 9782,16H  TONE TOO SMALL)
     GO TO 9991
9774 CONTINUE
 322 CONTINUE
     PRINT 563,CGUESS,TONE
 563 FORMAT (16H 563  NEW C IS  ,F20.10,15H  NEW TONE IS  ,F20.10)
 999 CONTINUE
     GO TO 1
 998 DTMT=XONE(2)*(CORA*XONE(5)-CORB*YONE(1))
   +     +XONE(1)*(CORB*CORR(2)-CORA*CORR(1))
     IF(ABSF(DTMT)-.01)9601,9601,9602
9601 IF(ABSF(DTMT)-.00001)9603,9603,9604
9603 PRINT 9651,XBND
9651 FORMAT(5H 9651,9H  XMAX=  ,F20.10)
     GO TO 9991
9604 CONTINUE
     GO TO 9602
9602 EPS2=EPSFAC*DTMT
     DELTAX=XBND-X11
     IF(DELTAX)8401,8401,8402
8401 EPS=EPS*.1
```

```
      GO TO 9627
3402 EPS=EPS2
      X11=XBND
      GO TO 9616
9627 TONE=TONE-DELT1T
      DGUESS=DGUESS-DELT1D
      CGUESS=CGUESS-DELT1C
9616 DELT1C=EPS*(-XONE(1)*CORA)
      DELT1D=EPS*(XONE(1)*CORR(2)-XONE(2)*YONE(1))
      DELT1T=EPS*(XONE(2)*CORA)
6621 IF(ABSF(DELT1C)+ABSF(DELT1D)+ABSF(DELT1T)-.5)6622,6623,6623
6623 DELT1C=.5*DELT1C
      DELT1D=.5*DELT1D
      DELT1T=.5*DELT1T
      GO TO 6621
6622 CONTINUE
      PRINT 9611,DELT1C,DELT1D,DELT1T
9611 FORMAT(5H 9601,10H DELTAC=  ,F20.10,11H  DELTAD=  ,F20.10,11H  DEL
    1TAT=  ,F20.10)
      CGUESS=CGUESS+DELT1C
      DGUESS=DGUESS+DELT1D
      TONE=TONE+DELT1T
      ATANC=ATANF(CGUESS)
      ATAND=ATANF(DGUESS)
      PRINT 9617,ATANC,ATAND
9617 FORMAT(5H 9617,6H  C=  ,F20.10,10H RADIANS, ,6H  D=  ,F20.10,9H  R
    +ADIANS)
      DELTC=ATANC-ATC11
      DELTD=ATAND-ATD11
      IF(ABSF(DELTC)+ABSF(DELTD)-1.E-6) 9981,9981,9982
9981 PRINT 9983
9983 FORMAT(5H 9983,49H  CORRECTION IS LESS THAN .000001   - WE ARE DON
    1E//)
      GO TO 9991
9982 ATC11=ATANC
      ATD11=ATAND
```

```
9985 CONTINUE
     PRINT 9612,EPS
9612 FORMAT(12H 9612 EPS=  ,F20.10)
     AAK(1,1)=BK(1,1)+CGUESS*BK(1,2)
     AAK(2,1)=BK(2,1)+CGUESS*BK(2,2)
     AAK(1,2)=BK(1,1)+DGUESS*BK(1,2)
     AAK(2,2)=BK(2,1)+DGUESS*BK(2,2)
     PRINT 9987
9987 FORMAT(5H 9987,18H  K* AT TONE MINUS,10X,15HK* AT TONE PLUS//)
     PRINT 9988,((AAK(I,J),J=1,2),I=1,2)
9988 FORMAT(10X,F12.5,14X,F12.5)
     PRINT 9989
9989 FORMAT(1H0,12H  XDOT MINUS,10X,9HXDOT PLUS)
     PRINT 9990,XDOT1,XDOT2,YDOT1,YDOT2
9990 FORMAT(5X,F12.5,10X,F12.5)
     GO TO 1
9198 PRINT 9199
9199 FORMAT(68H RANK OF FIRST CORRECTION MATRIX IS ONE - NO ADMISSIBLE
    1CURVES EXIST)
9991 CONTINUE
     STOP
     END
     END
```

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Two problems are presented which are linear on two adjacent intervals but not on their union. These problems are associated with the differential equation $\dot{X} = \begin{cases} AX + BF, & 0 < t < t_1 \\ CX + DF, & t_1 < t < T \end{cases}$, where X is the matrix $\begin{pmatrix} x \\ y \end{pmatrix}$, where F is a 2 x 1 matrix, and where A,B,C, and D are 2 x 2 matrices of functions of t. t_1 is a variable, hence the differential equation is non linear. Problems associated with this differential equation are called semi-linear.

In the first problem, a condition is found on t_1 and F which must be satisfied whenever x(T) is to be a maximum with y(T) fixed. In the second problem, conditions on F and t_1 are found which must be satisfied for x(T) to be a maximum for a fixed y(T) and for a fixed $x(t_1)$. A numerical routine is developed which yields successive approximations to the maximum value of x(T).

The basic theory of the methods is presented, and the problems are developed in the context of optimum control.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Optimum Control Non-linear Optimum Control Problems Semi-linear Optimum Control Problems Successive Approximation Technique Maximization with Non-linear Boundary Conditions						

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